Brief Notes For Math 3710

Afshin Ghoreishi

Spring 2022

Contents

Preface

This is a preliminary collection of brief notes and handouts for Math 3710. It is written to match the 6th edition of Powers textbook. This is not a replacement for your own course notes! However, if you print this version and bring it to class, you can add class notes to it (space has been alloted for this) to make a complete set of course notes. You can obtain this document from the website for this course in `http://faculty.weber.edu/aghoreishi/`.

Chapter -1

Handouts (Chapter 0 and Appendix in the Course Textbook)

The chapter zero of your textbook contains a review of ordinary differential equations and in the appendix you find mathematical references. Read them, as needed. This chapter contains the handouts for the course which includes textbook corrections and review of both ODE's and a very large collection of mathematical references.

-0.9 Handouts

A Partial List of Corrections to the
Boundary Value Problems
by David L. Powers, Sixth Edition

	Location	Original	Correction
1.	Chapter 1, Misc Exer 7(f)	$f(x) = x$	$f(x) = 0$
2.	Chapter 1, Misc Exer 8	$f(x) = 0$	$f(x) = x$
3.	Sec 2.1, Exer 3	What is $g \cdots$	What is the replacement of $A\Delta x g \cdots$
4.	Sec 3.2, Exer 12	$f(x)$ is as in Eq. (11)	$f(x)$ is as in the example in page 221.
5.	Sec 3.3, Exer 6	... of Exercise 3 as ...	... of Exercise 5 as ....
6.	Page 238, Equation (16)	dx_n	$d\,x$
7.	Sec 3.4, Exer 8	$\frac{\partial u^2}{\partial x^2}$ $u(x,\, t) = f(x)$	$\frac{\partial^2 u}{\partial x^2}$ $u(x,\, 0) = f(x)$
8.	Sec 4.5, Exer 8(e)	(... , Exercise 1)	(... , Exercise 5(a))
9.	Page 309, 2nd paragraph	$\Delta x(\cos(\delta) - \cos(\gamma))$	$\sigma\Delta x(\cos(\delta) - \cos(\gamma))$
10.	Page 311, 3rd equation	$\sigma\Delta y \left(\frac{\partial y}{\partial x}(x + \Delta x, y, t) \ldots\right.$	$\sigma\Delta y \left(\frac{\partial u}{\partial x}(x + \Delta x, y, t) \ldots\right.$
11.	Page 323, Equation (3)	$\phi(r,\, \pi)$	$\phi(r,\, \theta)$
12.	Page 381, Equation for $\mathcal{L}(f''(t))$	$= -f(0) + \cdots = -f(0) - \cdots$	$= -f'(0) + \cdots = -f'(0) - \cdots$
13.	Page 457, Solution to Exer 3, Sec 1.9	$\begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{for } 1 < x \end{cases}$	$\begin{cases} 1 & \text{for } 0 < \lambda < 1 \\ 0 & \text{for } 1 < \lambda \end{cases}$
14.	Page 466, Solution to Exer 3, Sec 2.10	Eq. (6)	Eq. (9)
15.	Page 471, Solution to Exer 5, Sec 3.3	$u(0.5a,\, 1.2a/c) = -0.2\alpha a$	$u(0.5a,\, 1.2a/c) = \alpha a/2$
16.	Page 490, Solution to Exer 1, Chap 5, Misc Exer	$\mu_m = m\pi b$	$\mu_m = m\pi/b$
17.	Page 491, Solution to Exer 3, Chap 5, Misc Exer	$\sum_{n=1}^{\infty} \cdots$	$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \cdots$

Review, Identities, Formulas and Theorems

Let n, $\overline{n}$, m, $\overline{m}$, k, $\overline{k}$, l, p, q and $\overline{q}$ be nonnegative integers, unless stated otherwise. Let z be a nonnegative real number, unless stated otherwise.

Trigonometric Identities

1. $\sin a \cos b = \frac{1}{2}[\sin(a+b) + \sin(a-b)]$ 2. $\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$
3. $\cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)]$

Hyperbolic Functions

4. $\sinh x = \frac{e^x - e^{-x}}{2}$ 5. $\cosh x = \frac{e^x + e^{-x}}{2}$ 6. $\tanh x = \frac{\sinh x}{\cosh x}$ 7. $\coth x = \frac{\cosh x}{\sinh x}$ 8. $\operatorname{sech} x = \frac{1}{\cosh x}$
9. $\operatorname{csch} x = \frac{1}{\sinh x}$

10. $\sinh(-x) = -\sinh x$ 11. $\cosh(-x) = \cosh x$ 12. $\cosh^2 x - \sinh^2 x = 1$ 13. $1 - \tanh^2 x = \operatorname{sech}^2 x$
14. $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$ 15. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

16. $\frac{d}{dx}(\sinh x) = \cosh x$ 17. $\frac{d}{dx}(\cosh x) = \sinh x$ 18. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$ 19. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
20. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$ 21. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

Integrals

22. $\displaystyle\int x \sin ax\, dx = \frac{1}{a^2}\sin ax - \frac{x}{a}\cos ax + C$ 23. $\displaystyle\int x \cos ax\, dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax + C$

24. $\displaystyle\int x^2 \sin ax\, dx = \frac{2}{a^3}\cos ax + \frac{2}{a^2}x \sin ax - \frac{1}{a}x^2 \cos ax + C$

25. $\displaystyle\int x^2 \cos ax\, dx = -\frac{2}{a^3}\sin ax + \frac{2}{a^2}x \cos ax + \frac{1}{a}x^2 \sin ax + C$

26. $\displaystyle\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + C$ 27. $\displaystyle\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) + C$

Definite Integrals

28. $\displaystyle\int_0^a \sin\frac{n\pi x}{a} \sin\frac{m\pi x}{a}\, dx = \begin{cases} \frac{a}{2}, & \text{if } n = m \neq 0 \\ 0, & \text{otherwise} \end{cases}$ 29. $\displaystyle\int_0^a \cos\frac{n\pi x}{a} \cos\frac{m\pi x}{a}\, dx = \begin{cases} a, & \text{if } n = m = 0 \\ \frac{a}{2}, & \text{if } n = m \neq 0 \\ 0, & \text{otherwise} \end{cases}$

30. For positive integers n and m, $\displaystyle\int_0^a \cos\frac{(2n-1)\pi x}{2a} \cos\frac{(2m-1)\pi x}{2a}\, dx = \begin{cases} \frac{a}{2}, & \text{if } n = m \\ 0, & \text{otherwise} \end{cases}$

31. For positive integers n and m, $\displaystyle\int_0^a \sin\frac{(2n-1)\pi x}{2a} \sin\frac{(2m-1)\pi x}{2a}\, dx = \begin{cases} \frac{a}{2}, & \text{if } n = m \\ 0, & \text{otherwise} \end{cases}$

32. $\displaystyle\int_{-a}^a \sin\frac{n\pi x}{a} \sin\frac{m\pi x}{a}\, dx = \begin{cases} a, & \text{if } n = m \neq 0 \\ 0, & \text{otherwise} \end{cases}$ 33. $\displaystyle\int_{-a}^a \cos\frac{n\pi x}{a} \cos\frac{m\pi x}{a}\, dx = \begin{cases} 2a, & \text{if } n = m = 0 \\ a, & \text{if } n = m \neq 0 \\ 0, & \text{otherwise} \end{cases}$

34. $\displaystyle\int_{-a}^a \sin\frac{n\pi x}{a} \cos\frac{m\pi x}{a}\, dx = 0$

35. For $0 < \alpha_1 < \alpha_2 < \cdots$ zeros of $J_z(x)$ and $z \geq 0$, $\displaystyle\int_0^a J_z(\tfrac{\alpha_m r}{a}) J_z(\tfrac{\alpha_{\overline{m}} r}{a})\, r\, dr = \begin{cases} \frac{a^2}{2} J_{z+1}^2(\alpha_m), & \text{if } \overline{m} = m \\ 0, & \text{otherwise} \end{cases}$

36. For $0 < \beta_1 < \beta_2 < \cdots$ zeros of $J_0'(x)$, $\displaystyle\int_0^a J_0(\frac{\beta_m r}{a})\, r\, dr = 0$ and

$$\int_0^a J_0(\tfrac{\beta_m r}{a}) J_0(\tfrac{\beta_{\overline{m}} r}{a})\, r\, dr = \begin{cases} \frac{a^2}{2} J_0^2(\beta_m), & \text{if } \overline{m} = m \\ 0, & \text{otherwise} \end{cases}$$

37. For $0 < \beta_1 < \beta_2 < \cdots$ zeros of $J_z'(x)$ and $z > 0$,

$$\int_0^a J_z(\tfrac{\beta_m r}{a}) J_z(\tfrac{\beta_{\overline{m}} r}{a})\, r\, dr = \begin{cases} \frac{a^2}{2}[J_z^2(\beta_m) - J_{z-1}(\beta_m) J_{z+1}(\beta_m)], & \text{if } \overline{m} = m \\ 0, & \text{otherwise} \end{cases}$$

38*. $$\int_0^\pi P_n^m(\cos\phi) P_{\overline{n}}^m(\cos\phi) \sin\phi\, d\phi = \begin{cases} \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}, & \text{if } \overline{n} = n \geq m \\ 0, & \text{otherwise} \end{cases}$$

39*. $$\int_{-1}^1 P_n^m(s) P_{\overline{n}}^m(s)\, ds = \begin{cases} \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}, & \text{if } \overline{n} = n \geq m \\ 0, & \text{otherwise} \end{cases}$$

* For $m = 0$, $P_k^m(s) = P_k(s)$ and $\frac{(n+m)!}{(n-m)!} = 1$.

Definite Double Integrals

40. $$\int_0^a \int_0^b \sin\tfrac{n\pi x}{a} \sin\tfrac{m\pi y}{b} \sin\tfrac{p\pi x}{a} \sin\tfrac{q\pi y}{b}\, dy\, dx = \begin{cases} \frac{ab}{4}, & \text{if } n = p \neq 0 \text{ and } m = q \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

41. $$\int_0^a \int_0^b \sin\tfrac{n\pi x}{a} \cos\tfrac{m\pi y}{b} \sin\tfrac{p\pi x}{a} \cos\tfrac{q\pi y}{b}\, dy\, dx = \begin{cases} \frac{ab}{2}, & \text{if } n = p \neq 0 \text{ and } m = q = 0 \\ \frac{ab}{4}, & \text{if } n = p \neq 0 \text{ and } m = q \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

42. $$\int_0^a \int_0^b \cos\tfrac{n\pi x}{a} \cos\tfrac{m\pi y}{b} \cos\tfrac{p\pi x}{a} \cos\tfrac{q\pi y}{b}\, dy\, dx = \begin{cases} ab, & \text{if } n = m = p = q = 0 \\ \frac{ab}{2}, & \text{if } n = p \neq 0 \text{ and } m = q = 0 \\ \frac{ab}{2}, & \text{if } n = p = 0 \text{ and } m = q \neq 0 \\ \frac{ab}{4}, & \text{if } n = p \neq 0 \text{ and } m = q \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

Suppose $0 < \alpha_1 < \alpha_2 < \cdots$ are zeros of $J_z(x)$.

43. $$\int_0^a \int_0^b J_z(\tfrac{\alpha_m r}{a}) J_z(\tfrac{\alpha_{\overline{m}} r}{a})\, r \cos\tfrac{k\pi\theta}{b} \cos\tfrac{\overline{k}\pi\theta}{b}\, d\theta\, dr = \begin{cases} \frac{a^2 b}{2} J_{z+1}^2(\alpha_m), & \text{if } \overline{m} = m \text{ and } \overline{k} = k = 0 \\ \frac{a^2 b}{4} J_{z+1}^2(\alpha_m), & \text{if } \overline{m} = m \text{ and } \overline{k} = k \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

44. $$\int_0^a \int_0^b J_z(\tfrac{\alpha_m r}{a}) J_z(\tfrac{\alpha_{\overline{m}} r}{a})\, r \sin\tfrac{k\pi\theta}{b} \sin\tfrac{\overline{k}\pi\theta}{b}\, d\theta\, dr = \begin{cases} \frac{a^2 b}{4} J_{z+1}^2(\alpha_m), & \text{if } \overline{m} = m \text{ and } \overline{k} = k \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

45. $$\int_0^a \int_{-b}^b J_z(\tfrac{\alpha_m r}{a}) J_z(\tfrac{\alpha_{\overline{m}} r}{a})\, r \cos\tfrac{k\pi\theta}{b} \cos\tfrac{\overline{k}\pi\theta}{b}\, d\theta\, dr = \begin{cases} a^2 b J_{z+1}^2(\alpha_m), & \text{if } \overline{m} = m \text{ and } \overline{k} = k = 0 \\ \frac{a^2 b}{2} J_{z+1}^2(\alpha_m), & \text{if } \overline{m} = m \text{ and } \overline{k} = k \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

46. $$\int_0^a \int_{-b}^b J_z(\tfrac{\alpha_m r}{a}) J_z(\tfrac{\alpha_{\overline{m}} r}{L})\, r \sin\tfrac{k\pi\theta}{b} \sin\tfrac{\overline{k}\pi\theta}{b}\, d\theta\, dr = \begin{cases} \frac{a^2 b}{2} J_{z+1}^2(\alpha_m), & \text{if } \overline{m} = m \text{ and } \overline{k} = k \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

47*. $\displaystyle\int_0^\pi \int_0^b P_n^m(\cos\phi)P_{\overline{n}}^m(\cos\phi)\sin\phi\cos\frac{k\pi\theta}{b}\cos\frac{\overline{k}\pi\theta}{b}\,d\theta\,d\phi = \begin{cases} \frac{2b}{2n+1}\frac{(n+m)!}{(n-m)!}, & \text{if } \overline{n}=n\ge m \text{ and } \overline{k}=k=0 \\ \frac{b}{2n+1}\frac{(n+m)!}{(n-m)!}, & \text{if } \overline{n}=n\ge m \text{ and } \overline{k}=k\ne 0 \\ 0, & \text{otherwise} \end{cases}$

48*. $\displaystyle\int_0^\pi \int_0^b P_n^m(\cos\phi)P_{\overline{n}}^m(\cos\phi)\sin\phi\sin\frac{k\pi\theta}{b}\sin\frac{\overline{k}\pi\theta}{b}\,d\theta\,d\phi = \begin{cases} \frac{b}{2n+1}\frac{(n+m)!}{(n-m)!}, & \text{if } \overline{n}=n\ge m \text{ and } \overline{k}=k\ne 0 \\ 0, & \text{otherwise} \end{cases}$

49*. $\displaystyle\int_0^\pi \int_{-b}^b P_n^m(\cos\phi)P_{\overline{n}}^m(\cos\phi)\sin\phi\cos\frac{k\pi\theta}{b}\cos\frac{\overline{k}\pi\theta}{b}\,d\theta\,d\phi = \begin{cases} \frac{4b}{2n+1}\frac{(n+m)!}{(n-m)!}, & \text{if } \overline{n}=n\ge m \text{ and } \overline{k}=k=0 \\ \frac{2b}{2n+1}\frac{(n+m)!}{(n-m)!}, & \text{if } \overline{n}=n\ge m \text{ and } \overline{k}=k\ne 0 \\ 0, & \text{otherwise} \end{cases}$

50*. $\displaystyle\int_0^\pi \int_{-b}^b P_n^m(\cos\phi)P_{\overline{n}}^m(\cos\phi)\sin\phi\sin\frac{k\pi\theta}{b}\sin\frac{\overline{k}\pi\theta}{b}\,d\theta\,d\phi = \begin{cases} \frac{2b}{2n+1}\frac{(n+m)!}{(n-m)!}, & \text{if } \overline{n}=n\ge m \text{ and } \overline{k}=k\ne 0 \\ 0, & \text{otherwise} \end{cases}$

* It also holds if $k = m$. For $m = 0$, $P_k^m(s) = P_k(s)$ and $\frac{(n+m)!}{(n-m)!} = 1$.

Definite Triple Integrals

Suppose $0 < \alpha_1 < \alpha_2 < \cdots$ are zeros of $J_z(x)$.

51**. $\displaystyle\int_0^a \int_0^\pi \int_0^b J_z(\tfrac{\alpha_m\rho}{a})J_z(\tfrac{\alpha_{\overline{m}}\rho}{a})\,\rho\,P_k^l(\cos\phi)P_{\overline{k}}^l(\cos\phi)\sin\phi\cos\frac{q\pi\theta}{b}\cos\frac{\overline{q}\pi\theta}{b}\,d\theta\,d\phi\,d\rho =$

$$\begin{cases} \frac{a^2b}{2k+1}\frac{(k+l)!}{(k-l)!}J_{z+1}^2(\alpha_m), & \text{if } \overline{m}=m,\ \overline{k}=k\ge l \text{ and } \overline{q}=q=0 \\ \frac{a^2b}{2(2k+1)}\frac{(k+l)!}{(k-l)!}J_{z+1}^2(\alpha_m), & \text{if } \overline{m}=m,\ \overline{k}=k\ge l \text{ and } \overline{q}=q\ne 0 \\ 0, & \text{otherwise} \end{cases}$$

52**. $\displaystyle\int_0^a \int_0^\pi \int_0^b J_z(\tfrac{\alpha_m\rho}{a})J_z(\tfrac{\alpha_{\overline{m}}\rho}{a})\,\rho\,P_k^l(\cos\phi)P_{\overline{k}}^l(\cos\phi)\sin\phi\sin\frac{q\pi\theta}{b}\sin\frac{\overline{q}\pi\theta}{b}\,d\theta\,d\phi\,d\rho =$

$$\begin{cases} \frac{a^2b}{2(2k+1)}\frac{(k+l)!}{(k-l)!}J_{z+1}^2(\alpha_m), & \text{if } \overline{m}=m,\ \overline{k}=k\ge l \text{ and } \overline{q}=q\ne 0 \\ 0, & \text{otherwise} \end{cases}$$

53**. $\displaystyle\int_0^a \int_0^\pi \int_{-b}^b J_z(\tfrac{\alpha_m\rho}{a})J_z(\tfrac{\alpha_{\overline{m}}\rho}{a})\,\rho\,P_k^l(\cos\phi)P_{\overline{k}}^l(\cos\phi)\sin\phi\cos\frac{q\pi\theta}{b}\cos\frac{\overline{q}\pi\theta}{b}\,d\theta\,d\phi\,d\rho =$

$$\begin{cases} \frac{2a^2b}{2k+1}\frac{(k+l)!}{(k-l)!}J_{z+1}^2(\alpha_m), & \text{if } \overline{m}=m,\ \overline{k}=k\ge l \text{ and } \overline{q}=q=0 \\ \frac{a^2b}{2k+1}\frac{(k+l)!}{(k-l)!}J_{z+1}^2(\alpha_m), & \text{if } \overline{m}=m,\ \overline{k}=k\ge l \text{ and } \overline{q}=q\ne 0 \\ 0, & \text{otherwise} \end{cases}$$

54**. $\displaystyle\int_0^a \int_0^\pi \int_{-b}^b J_z(\tfrac{\alpha_m\rho}{a})J_z(\tfrac{\alpha_{\overline{m}}\rho}{a})\,\rho\,P_k^l(\cos\phi)P_{\overline{k}}^l(\cos\phi)\sin\phi\sin\frac{q\pi\theta}{b}\sin\frac{\overline{q}\pi\theta}{b}\,d\theta\,d\phi\,d\rho =$

$$\begin{cases} \frac{a^2b}{2k+1}\frac{(k+l)!}{(k-l)!}J_{z+1}^2(\alpha_m), & \text{if } \overline{m}=m,\ \overline{k}=k\ge l \text{ and } \overline{q}=q\ne 0 \\ 0, & \text{otherwise} \end{cases}$$

** For $l = 0$, $P_n^l(s) = P_n(s)$ and $\frac{(k+l)!}{(k-1)!} = 1$.

Ordinary Differential Equations

55. First Order Linear ODE: $y' + f(x)y = g(x)$

Integrating Factor: $\mu(x) = e^{\int f(x)dx}$ with $C = 0$, $\mu(x)y'(x) + \mu(x)f(x)y(x) = \mu(x)g(x) \Longrightarrow$

$\frac{d}{dx}[\mu(x)y(x)] = \mu(x)g(x) \Longrightarrow \mu(x)y(x) = \displaystyle\int \mu(x)g(x)\,dx + C \Longrightarrow y(x) = \frac{1}{\mu(x)}\int \mu(x)g(x)\,dx + \frac{C}{\mu(x)}$

Or, integrating factor: $\mu(x) = e^{\int_{x_0}^x f(t)dt}$ and $y(x) = \dfrac{1}{\mu(x)}\displaystyle\int_{x_0}^x \mu(t)g(t)\,dt + \frac{y(x_0)}{\mu(x)}$

56. First Order Separable ODE: $\dfrac{dy}{dx} = \dfrac{g(x)}{h(y)}$

Implicit Solution: $\displaystyle\int h(y)\,dy = \int g(x)\,dx \Longrightarrow H(y) = G(x) + C$ with $H' = h$ and $G' = g$

Or, $\displaystyle\int_{y(x_0)}^{y} h(t)\,dt = \int_{x_0}^{x} g(t)\,dt$

57. Exact ODE: $M(x,\,y) + N(x,\,y)\dfrac{dy}{dx} = 0$ is called exact if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Implicit Solution: $F(x,\,y) = C$ where $\dfrac{\partial F}{\partial x} = M$ and $\dfrac{\partial F}{\partial y} = N$

Start with $\frac{\partial F}{\partial x} = M$ or $\frac{\partial F}{\partial y} = N$ integrate with respect to x or y, respectively, then differentiate with respect to the other variable, and use the other equation to find the remaining function of y or x.

58. Second Order Linear ODE with Constant Coefficients: $ay'' + by' + cy = 0$

Characteristic Equation: $ar^2 + br + c = 0$ with solutions r_1 and r_2

$$y(x) = \begin{cases} c_1e^{r_1x} + c_2e^{r_2x}, & \text{if } r_1 \text{ and } r_2 \text{ are real-valued and unequal} \\ c_1e^{r_1x} + c_2\,x\,e^{r_1x}, & \text{if } r_1 = r_2 \\ c_1e^{\lambda x}\cos\mu x + c_2e^{\lambda x}\sin\mu x, & \text{if } r_1,\, r_2 = \lambda \pm \mu i \end{cases}$$

If $r_1,\, r_2 = \pm r$, then $y(x) = c_1e^{-rx} + c_2e^{rx}$ or $y(x) = c_1\cosh rx + c_2\sinh rx$ or

$y(x) = c_1\cosh r(x - x_0) + c_2\sinh r(x - x_0)$ or

$y(x) = c_1\sinh r(x - x_0) + c_2\sinh rx$ or $y(x) = c_1\cosh r(x - x_0) + c_2\cosh rx$

59. Second Order Linear Nonhomogeneous ODE: $y'' + p(x)y' + q(x)y = g(x)$

General Solution: $y(x) = y_h(x) + y_p(x)$ where the homogeneous solution $y_h(x) = c_1y_1(x) + c_2y_2(x)$ is the general solution to the homogeneous equation $y'' + p(x)y' + q(x)y = 0$, while y_1 and y_2 are two linearly independent solutions of the same homogeneous equation, and the particular solution $y_p(x)$ is a solution to the nonhomogeneous equation $y'' + p(x)y' + q(x)y = g(x)$.

Method of Variation of Parameters: $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ where $u_1'(x) = \frac{-y_2(x)g(x)}{W(x)}$, $u_2'(x) = \frac{y_1(x)g(x)}{W(x)}$ and the Wronskian $W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$.

$$y_p(x) = y_1(x)\int \frac{-y_2(x)g(x)}{W(x)}\,dx + y_2(x)\int \frac{y_1(x)g(x)}{W(x)}\,dx \text{ or}$$

$$y_p(x) = y_1(x)\int_{x_0}^{x} \frac{-y_2(t)g(t)}{W(t)}\,dt + y_2(x)\int_{x_0}^{x} \frac{y_1(t)g(t)}{W(t)}\,dt$$

60. Cauchy-Euler Equation: $x^2y'' + \alpha xy' + \beta y = 0$

Indicial Equation: $p(p-1) + \alpha p + \beta = 0$ with solutions p_1 and p_2

$$y(x) = \begin{cases} c_1|x|^{p_1} + c_2|x|^{p_2}, & \text{if } p_1 \text{ and } p_2 \text{ are real-valued and unequal} \\ (c_1 + c_2\ln|x|)|x|^{p_1}, & \text{if } p_1 = p_2 \\ |x|^{\lambda}[c_1\cos(\mu\ln|x|) + c_2\sin(\mu\ln|x|)], & \text{if } p_1,\, p_2 = \lambda \pm \mu i \end{cases}$$

61. $x^2\dfrac{d^2\phi}{dx^2} + x\dfrac{d\phi}{dx} - n^2\phi = 0$ and $\phi(0)$ bounded $\Longrightarrow \phi(x) = x^n$ for $n = 0\,,\,1\,,\cdots$

Rayleigh Quotients

62. $\dfrac{d}{dx}\left[s(x)\,\frac{d\phi}{dx}\right] - q(x)\phi + \lambda p(x)\phi = 0 \Longrightarrow \lambda = \dfrac{-s(x)\phi(x)\,\frac{d\phi}{dx}\Big|_l^r + \displaystyle\int_l^r \left[s(x)(\frac{d\phi}{dx})^2 + q(x)\phi^2(x)\right]\,dx}{\displaystyle\int_l^r \phi^2(x)p(x)\,dx}$

63. $\nabla^2\phi + \lambda\phi = 0 \Longrightarrow \lambda = \dfrac{-\displaystyle\oint_C \phi\nabla\phi\cdot\hat{n}\,ds + \iint_R |\nabla\phi|^2\,dA}{\displaystyle\iint_R \phi^2\,dA}$

Lagrange's Identity and Green's Formula

For $L(\phi) = \dfrac{d}{dx}\left[s(x)\,\dfrac{d\phi}{dx}\right] - q(x)\phi,$

64. $uL(v) - vL(u) = \dfrac{d}{dx}\left[s(x)\left(u(x)\dfrac{dv}{dx} - v(x)\dfrac{du}{dx}\right)\right]$

65. $\displaystyle\int_l^r [uL(v) - vL(u)]\,dx = s(x)\,\left[u(x)\frac{dv}{dx} - v(x)\frac{du}{dx}\right]\Bigg|_l^r$

Green's Identities

66. $\displaystyle\iint_R u\nabla^2 v\,dA = \oint_C u\nabla v\cdot\hat{n}\,ds - \iint_R \nabla u\cdot\nabla v\,dA$

67. $\displaystyle\iint_R (u\nabla^2 v - v\nabla^2 u)\,dA = \oint_C (u\nabla v - v\nabla u)\cdot\hat{n}\,ds$

68. $\displaystyle\iiint_\Omega (u\nabla^2 v - v\nabla^2 u)\,dV = \oiint_{\partial\Omega} (u\nabla v - v\nabla u)\cdot\hat{n}\,dS$

Eigenvalue Problems

69. $\dfrac{d^2\phi}{dx^2} = -\lambda\phi\,,\ \phi(0) = 0$ and $\phi(a) = 0 \Longrightarrow \lambda = (\frac{n\pi}{a})^2\,,\ \phi(x) = \sin\frac{n\pi x}{a}$ for $n = 1\,,\,2\,,\cdots$

70. $\dfrac{d^2\phi}{dx^2} = -\lambda\phi\,,\ \dfrac{d\phi}{dx}(0) = 0$ and $\dfrac{d\phi}{dx}(a) = 0 \Longrightarrow \lambda = (\frac{n\pi}{a})^2\,,\ \phi(x) = \cos\frac{n\pi x}{a}$ for $n = 0\,,\,1\,,\cdots$

71. $\left\{\begin{array}{l} \dfrac{d^2\phi}{dx^2} = -\lambda\phi \\ \phi(-a) = \phi(a) \\ \dfrac{d\phi}{dx}(-a) = \dfrac{d\phi}{dx}(a) \end{array}\right. \Longrightarrow \begin{array}{l} \lambda = (\frac{n\pi}{a})^2 \\ \phi(x) = \cos\frac{n\pi x}{a} \text{ and } \sin\frac{n\pi x}{a} \end{array}$ for $n = 0\,,\,1\,,\cdots$

72. $\left\{\begin{array}{l} \dfrac{d^2\phi}{dx^2} = -\lambda\phi \\ \phi(0) = 0 \\ \dfrac{d\phi}{dx}(a) = 0 \end{array}\right. \Longrightarrow \begin{array}{l} \lambda = [\frac{(2n-1)\pi}{2a}]^2 \\ \phi(x) = \sin\frac{(2n-1)\pi x}{2a} \end{array}$ for $n = 1\,,\,2\,,\cdots$

73. $\left\{\begin{array}{l} \dfrac{d^2\phi}{dx^2} = -\lambda\phi \\ \dfrac{d\phi}{dx}(0) = 0 \\ \phi(a) = 0 \end{array}\right. \Longrightarrow \begin{array}{l} \lambda = [\frac{(2n-1)\pi}{2a}]^2 \\ \phi(x) = \cos\frac{(2n-1)\pi x}{2a} \end{array}$ for $n = 1\,,\,2\,,\cdots$

74. $$\begin{cases} x^2\dfrac{d^2\phi}{dx^2} + x\dfrac{d\phi}{dx} + (\lambda x^2 - n^2)\phi = 0 \\ \phi(0) \text{ bounded} \\ \phi(a) = 0 \end{cases} \implies \begin{array}{l} \lambda = (\frac{\alpha_m}{a})^2 \\ \phi(x) = J_n(\frac{\alpha_m x}{a}) \end{array} \quad \text{for } 0 < \alpha_1 < \alpha_2 < \cdots \text{ zeros of } J_n(x)$$

75. $$\begin{cases} \dfrac{d}{d\rho}\left[\rho^2 \dfrac{df}{d\rho}\right] + [\lambda\rho^2 - n(n+1)]f(\rho) = 0 \\ f(0) \text{ bounded} \\ f(a) = 0 \end{cases} \implies \begin{array}{l} \lambda = (\frac{\alpha_k}{a})^2 \\ f(\rho) = \rho^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\frac{\alpha_k \rho}{a}) \\ \text{for } 0 < \alpha_1 < \alpha_2 < \cdots \text{ zeros of } J_{n+\frac{1}{2}}(\rho) \end{array}$$

76. $$\begin{cases} x^2\dfrac{d^2\phi}{dx^2} + x\dfrac{d\phi}{dx} + (\lambda x^2 - n^2)\phi = 0 \\ \phi(0) \text{ bounded} \\ \dfrac{d\phi}{dx}(a) = 0 \end{cases} \implies \begin{array}{l} n = 0\,,\ \lambda = 0\,,\ \phi(x) = 1 \\ n > 0\,,\ \lambda = (\frac{\beta_m}{a})^2\,,\ \phi(x) = J_n(\frac{\beta_m x}{a}) \\ \text{for } 0 < \beta_1 < \beta_2 < \cdots \text{ zeros of } J_n'(x) \end{array}$$

77*. $$\begin{cases} \dfrac{d}{d\phi}\left[\sin\phi \dfrac{dg}{d\phi}\right] + \left(-\mu \sin\phi - \dfrac{m^2}{\sin\phi}\right) g(\phi) = 0 \\ g(0) \text{ and } g(\pi) \text{ bounded} \end{cases} \implies \begin{array}{l} \mu = -n(n+1) \\ g(\phi) = P_n^m(\cos\phi) \end{array} \quad \text{for } n = m\,,\, m+1\,,\cdots$$

78*. $$\begin{cases} (1-s^2)\dfrac{d^2\phi}{ds^2} - 2s\dfrac{d\phi}{ds} + \left(-\mu - \frac{m^2}{1-s^2}\right)\phi = 0 \\ \phi(-1) \text{ and } \phi(1) \text{ bounded} \end{cases} \implies \begin{array}{l} \mu = -n(n+1) \\ \phi(s) = P_n^m(s) \end{array} \quad \text{for } n = m\,,\, m+1\,,\cdots$$

* For $m = 0$, $P_n^m(s) = P_n(s)$.

Two-Dimensional Eigenvalue Problems

79. $$\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = -\lambda\phi(x,\,y) \\ \phi(0,\,y) = \phi(a,\,y) = 0 \\ \phi(x,\,0) = \phi(x,\,b) = 0 \end{cases} \implies \begin{array}{l} \lambda = (\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2 \\ \phi(x) = \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b} \end{array} \quad \text{for } n = 1\,,\,2\,,\cdots \text{ and } m = 1\,,\,2\,,\cdots$$

80. $$\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = -\lambda\phi(x,\,y) \\ \dfrac{\partial\phi}{\partial x}(0,\,y) = \dfrac{\partial\phi}{\partial x}(a,\,y) = 0 \\ \dfrac{\partial\phi}{\partial y}(x,\,0) = \dfrac{\partial\phi}{\partial y}(x,\,b) = 0 \end{cases} \implies \begin{array}{l} \lambda = (\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2 \\ \phi(x) = \cos\frac{n\pi x}{a}\cos\frac{m\pi y}{b} \end{array} \quad \text{for } n = 0\,,\,1\,,\cdots \text{ and } m = 0\,,\,1\,,\cdots$$

81. $$\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = -\lambda\phi(x,\,y) \\ \phi(0,\,y) = \phi(a,\,y) = 0 \\ \dfrac{\partial\phi}{\partial y}(x,\,0) = \dfrac{\partial\phi}{\partial y}(x,\,b) = 0 \end{cases} \implies \begin{array}{l} \lambda = (\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2 \\ \phi(x) = \sin\frac{n\pi x}{a}\cos\frac{m\pi y}{b} \end{array} \quad \text{for } n = 1\,,\,2\,,\cdots \text{ and } m = 0\,,\,1\,,\cdots$$

82. $$\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = -\lambda\phi(x,\,y) \\ \dfrac{\partial\phi}{\partial x}(0,\,y) = \dfrac{\partial\phi}{\partial x}(a,\,0) = 0 \\ \phi(x,\,0) = \phi(x,\,b) = 0 \end{cases} \implies \begin{array}{l} \lambda = (\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2 \\ \phi(x) = \cos\frac{n\pi x}{a}\sin\frac{m\pi y}{b} \end{array} \quad \text{for } n = 0\,,\,1\,,\cdots \text{ and } m = 1\,,\,2\,,\cdots$$

83. $$\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = -\lambda\phi(x,\,y) \\ \phi(0,\,y) = \phi(a,\,y) = 0 \\ \dfrac{\partial\phi}{\partial y}(x,\,0) = \phi(x,\,b) = 0 \end{cases} \implies \begin{array}{l} \lambda = (\frac{n\pi}{a})^2 + [\frac{(2m-1)\pi}{2b}]^2 \\ \phi(x) = \sin\frac{n\pi x}{a}\cos\frac{(2m-1)\pi y}{2b} \end{array} \quad \text{for } n = 1\,,\,2\,,\cdots \text{ and } m = 1\,,\,2\,,\cdots$$

84. $\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = -\lambda\phi(x,\, y) \\ \phi(0,\, y) = \phi(a,\, y) = 0 \\ \phi(x,\, 0) = \frac{\partial\phi}{\partial y}(x,\, b) = 0 \end{cases} \Longrightarrow \begin{array}{l} \lambda = (\frac{n\pi}{a})^2 + [\frac{(2m-1)\pi}{2b}]^2 \\ \phi(x) = \sin\frac{n\pi x}{a}\sin\frac{(2m-1)\pi y}{2b} \end{array}$ for $n = 1\,,\, 2\,,\cdots$ and $m = 1\,,\, 2\,,\cdots$

85. $\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = -\lambda\phi(x,\, y) \\ \dfrac{\partial\phi}{\partial x}(0,\, y) = \dfrac{\partial\phi}{\partial x}(a,\, y) = 0 \\ \dfrac{\partial\phi}{\partial y}(x,\, 0) = \phi(x,\, b) = 0 \end{cases} \Longrightarrow \begin{array}{l} \lambda = (\frac{n\pi}{a})^2 + [\frac{(2m-1)\pi}{2b}]^2 \\ \phi(x) = \cos\frac{n\pi x}{a}\cos\frac{(2m-1)\pi y}{2b} \end{array}$ for $n = 0\,,\, 1\,,\cdots$ and $m = 1\,,\, 2\,,\cdots$

86. $\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = -\lambda\phi(x,\, y) \\ \dfrac{\partial\phi}{\partial x}(0,\, y) = \dfrac{\partial\phi}{\partial x}(a,\, y) = 0 \\ \phi(x,\, 0) = \dfrac{\partial\phi}{\partial y}(x,\, b) = 0 \end{cases} \Longrightarrow \begin{array}{l} \lambda = (\frac{n\pi}{a})^2 + [\frac{(2m-1)\pi}{2b}]^2 \\ \phi(x) = \cos\frac{n\pi x}{a}\sin\frac{(2m-1)\pi y}{2b} \end{array}$ for $n = 0\,,\, 1\,,\cdots$ and $m = 1\,,\, 2\,,\cdots$

87. $\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = -\lambda\phi(x,\, y) \\ \dfrac{\partial\phi}{\partial x}(0,\, y) = \phi(a,\, y) = 0 \\ \phi(x,\, 0) = \dfrac{\partial\phi}{\partial y}(x,\, b) = 0 \end{cases} \Longrightarrow \begin{array}{l} \lambda = [\frac{(2n-1)\pi}{2a}]^2 + [\frac{(2m-1)\pi}{2b}]^2 \\ \phi(x) = \cos\frac{(2n-1)\pi x}{2a}\sin\frac{(2m-1)\pi y}{2b} \end{array}$ for $n = 1\,,\, 2\,,\cdots$ and $m = 1\,,\, 2\,,\cdots$

88. $\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = -\lambda\phi(x,\, y) \\ \dfrac{\partial\phi}{\partial x}(0,\, y) = \phi(a,\, y) = 0 \\ \dfrac{\partial\phi}{\partial y}(x,\, 0) = \phi(x,\, b) = 0 \end{cases} \Longrightarrow \begin{array}{l} \lambda = [\frac{(2n-1)\pi}{2a}]^2 + [\frac{(2m-1)\pi}{2b}]^2 \\ \phi(x) = \cos\frac{(2n-1)\pi x}{2a}\cos\frac{(2m-1)\pi y}{2b} \end{array}$ for $n = 1\,,\, 2\,,\cdots$ and $m = 1\,,\, 2\,,\cdots$

89. $\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = -\lambda\phi(x,\, y) \\ \phi(0,\, y) = \dfrac{\partial\phi}{\partial x}(a,\, y) = 0 \\ \phi(x,\, 0) = \dfrac{\partial\phi}{\partial y}(x,\, b) = 0 \end{cases} \Longrightarrow \begin{array}{l} \lambda = [\frac{(2n-1)\pi}{2a}]^2 + [\frac{(2m-1)\pi}{2b}]^2 \\ \phi(x) = \sin\frac{(2n-1)\pi x}{2a}\sin\frac{(2m-1)\pi y}{2b} \end{array}$ for $n = 1\,,\, 2\,,\cdots$ and $m = 1\,,\, 2\,,\cdots$

Supporting Theorems

90. Green's Theorem (vector version)

Let R be a region in $\Re^2$ bounded by a piecewise-smooth, simple closed curve C with counterclockwise orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region containing R, then $\displaystyle\iint\limits_R \nabla\cdot\vec{F}\,dA = \oint\limits_C \vec{F}\cdot\hat{n}\,ds\,.$

91. Divergence Theorem

Let Ω be a simple solid region in $\Re^3$ and let $\partial\Omega$ be its boundary with the outward orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region containing Ω, then $\displaystyle\iiint\limits_\Omega \nabla\cdot\vec{F}\,dV = \oiint\limits_{\partial\Omega} \vec{F}\cdot\hat{n}\,dS\,.$

92. If function f is continuous, $f(x) \not\equiv 0$ and $f(x) \geq 0$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)\,dx > 0\,.$

93. For a continuous nonnegative function f if $\int_a^b f(x)\,dx = 0$, then $f(x) = 0$ for $a \le x \le b$.

94. Uniform Convergence Definition

    The sequence of functions $f_n : D \longrightarrow \Re$, $n = 1,\ 2, \cdots$, is said to converge uniformly to the function $f : D \longrightarrow \Re$ if for every $\epsilon > 0$, there is a natural number N such that for all $x \in D$ we have $|f_n(x) - f(x)| < \epsilon$ for all $n \ge N$.

95. Weierstrass M Test (A test for uniform convergence.)

    Suppose for each function $f_n : D \longrightarrow \Re$, $n = 1,\ 2, \cdots$, there exists a constant M_n with $|f_n(x)| \le M_n$ for all $x \in D$, and $\sum_{n=1}^{\infty} M_n$ converges. Then $\sum_{n=1}^{\infty} f_n$ converges uniformly.

96. Interchanging Limit and Integral

    Suppose functions $f_n : [a,\, b] \longrightarrow \Re$, $n = 1,\ 2, \cdots$, are continuous and converge uniformly to a function $f : [a,\, b] \longrightarrow \Re$. Then $\lim_{n\to\infty} \left[\int_a^b f_n(x)\,dx\right] = \int_a^b \left[\lim_{n\to\infty} f_n(x)\right] dx = \int_a^b f(x)\,dx$.

97. Interchanging Integral and Summation

    Suppose functions $f_n : [a,\, b] \longrightarrow \Re$, $n = 1,\ 2, \cdots$, are continuous, and $\sum_{n=1}^{\infty} f_n$ converges uniformly.

    Then $\int_a^b \left[\sum_{n=1}^{\infty} f_n(x)\right] dx = \sum_{n=1}^{\infty} \left[\int_a^b f(x)\,dx\right]$.

98. Interchanging Differentiation and Summation

    Suppose functions f_n, $n = 1,\ 2, \cdots$, are continuously differentiable, $\sum_{n=1}^{\infty} f_n$ converges pointwise, and $\sum_{n=1}^{\infty} f_n'$ converges uniformly. Then $\frac{d}{dx}\left[\sum_{n=1}^{\infty} f_n(x)\right] = \sum_{n=1}^{\infty}\left[\frac{d}{dx} f_n(x)\right]$.

99. Leibniz Integral Rule (Interchanging differentiation and integration with respect to different variables.)

    Suppose functions $f(x,\, y)$ and $\frac{\partial f}{\partial y}(x,\, y)$ are continuous on $[a,\, b] \times [c,\, d]$. Then

    $\frac{d}{dy}\left[\int_a^b f(x,\, y)\,dx\right] = \int_a^b \left[\frac{\partial}{\partial y} f(x,\, y)\right] dx.$

Fourier Series

100. If $f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{a}$ for $0 < x < a$, then $A_0 = \frac{1}{a}\int_0^a f(x)\,dx$ and $A_n = \frac{2}{a}\int_0^a f(x) \cos \frac{n\pi x}{a}\,dx$

101. If $f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{a}$ for $0 < x < a$, then $B_n = \frac{2}{a}\int_0^a f(x) \sin \frac{n\pi x}{a}\,dx$

102. If $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{a} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a}$ for $-a < x < a$, then $a_0 = \frac{1}{2a}\int_{-a}^a f(x)\,dx$,

     $a_n = \frac{1}{a}\int_{-a}^a f(x) \cos \frac{n\pi x}{a}\,dx$ and $b_n = \frac{1}{a}\int_{-a}^a f(x) \sin \frac{n\pi x}{a}\,dx$

103. If $f(x) = \sum_{n=1}^{\infty} A_n \cos \frac{(2n-1)\pi x}{2a}$ for $0 < x < a$, then $A_n = \frac{2}{a}\int_0^a f(x) \cos \frac{(2n-1)\pi x}{2a}\,dx$

104. If $f(x)=\sum_{n=1}^{\infty} B_n \sin\frac{(2n-1)\pi x}{2a}$ for $0<x<a$, then $B_n=\frac{2}{a}\int_0^a f(x)\sin\frac{(2n-1)\pi x}{2a}\,dx$

Generalized Fourier Series

105. Suppose $0<\beta_1<\beta_2<\cdots$ are zeros of $J_0'(x)$.

If $f(r)=a_0+\sum_{m=1}^{\infty} a_m J_0(\frac{\beta_m r}{a})$ for $0<r<a$, then $a_0=\frac{2}{a^2}\int_0^a f(r)\,r\,dr$ and

$$a_m=\frac{\int_0^a f(r)J_0(\frac{\beta_m r}{a})r\,dr}{\int_0^a J_0^2(\frac{\beta_m r}{a})r\,dr}=\frac{2\int_0^a f(r)J_0(\frac{\beta_m r}{a})r\,dr}{a^2J_0^2(\beta_m)}$$

106. Suppose $0<\beta_1<\beta_2<\cdots$ are zeros of $J_z'(x)$ and $z>0$.

If $f(r)=\sum_{m=1}^{\infty} a_m J_z(\frac{\beta_m r}{a})$ for $0<r<a$, then

$$a_m=\frac{\int_0^a f(r)J_z(\frac{\beta_m r}{a})r\,dr}{\int_0^a J_z^2(\frac{\beta_m r}{a})r\,dr}=\frac{2\int_0^a f(r)J_z(\frac{\beta_m r}{a})r\,dr}{a^2[J_z^2(\beta_m)-J_{z-1}(\beta_m)J_{z+1}(\beta_m)]}$$

107. If $f(\phi)=\sum_{n=m}^{\infty} a_n P_n^m(\cos\phi)$ for $0<\phi<\pi$ and $m>0$, then

$$a_n=\frac{\int_0^\pi f(\phi)P_n^m(\cos\phi)\sin\phi\,d\phi}{\int_0^\pi [P_n^m(\cos\phi)]^2\sin\phi\,d\phi}=\frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}\int_0^\pi f(\phi)P_n^m(\cos\phi)\sin\phi\,d\phi$$

108. If $f(\phi)=\sum_{n=0}^{\infty} a_n P_n(\cos\phi)$ for $0<\phi<\pi$, then

$$a_n=\frac{\int_0^\pi f(\phi)P_n(\cos\phi)\sin\phi\,d\phi}{\int_0^\pi [P_n(\cos\phi)]^2\sin\phi\,d\phi}=\frac{2n+1}{2}\int_0^\pi f(\phi)P_n(\cos\phi)\sin\phi\,d\phi$$

109. If $f(\phi)=\sum_{k=0}^{\infty} a_k P_{2k}(\cos\phi)$ for $0<\phi<\frac{\pi}{2}$, then

$$a_k=\frac{\int_0^{\frac{\pi}{2}} f(\phi)P_{2k}(\cos\phi)\sin\phi\,d\phi}{\int_0^{\frac{\pi}{2}} [P_{2k}(\cos\phi)]^2\sin\phi\,d\phi}=(4k+1)\int_0^{\frac{\pi}{2}} f(\phi)P_{2k}(\cos\phi)\sin\phi\,d\phi$$

110. If $f(\phi)=\sum_{k=1}^{\infty} a_k P_{2k-1}(\cos\phi)$ for $0<\phi<\frac{\pi}{2}$, then

$$a_k=\frac{\int_0^{\frac{\pi}{2}} f(\phi)P_{2k-1}(\cos\phi)\sin\phi\,d\phi}{\int_0^{\frac{\pi}{2}}[P_{2k-1}(\cos\phi)]^2\sin\phi\,d\phi}=(4k-1)\int_0^{\frac{\pi}{2}} f(\phi)P_{2k-1}(\cos\phi)\sin\phi\,d\phi$$

Double Fourier Series

111. If $f(x,\,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}$ for $(x,\,y)\in(0,\,a)\times(0,\,b)$, then

$$B_{nm}=\tfrac{4}{ab}\int_0^a\int_0^b f(x,\,y)\sin\tfrac{n\pi x}{a}\sin\tfrac{m\pi y}{b}\,dy\,dx$$

112. If $f(x,\,y)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{nm}\cos\frac{n\pi x}{a}\cos\frac{m\pi y}{b}$ for $(x,\,y)\in(0,\,a)\times(0,\,b)$, then

$$A_{00}=\tfrac{1}{ab}\int_0^a\int_0^b f(x,\,y)\,dy\,dx\,,\ A_{n0}=\tfrac{2}{ab}\int_0^a\int_0^b f(x,\,y)\cos\tfrac{n\pi x}{a}\,dy\,dx\,,$$

$$A_{0m}=\tfrac{2}{ab}\int_0^a\int_0^b f(x,\,y)\cos\tfrac{m\pi y}{b}\,dy\,dx \text{ and } A_{nm}=\tfrac{4}{ab}\int_0^a\int_0^b f(x,\,y)\cos\tfrac{n\pi x}{a}\cos\tfrac{m\pi y}{b}\,dy\,dx$$

113. If $f(x,\,y)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} C_{nm}\sin\frac{n\pi x}{a}\cos\frac{m\pi y}{b}$ for $(x,\,y)\in(0,\,a)\times(0,\,b)$, then

$$C_{n0}=\tfrac{2}{ab}\int_0^a\int_0^b f(x,\,y)\sin\tfrac{n\pi x}{a}\,dy\,dx \text{ and } C_{nm}=\tfrac{4}{ab}\int_0^a\int_0^b f(x,\,y)\sin\tfrac{n\pi x}{a}\cos\tfrac{m\pi y}{b}\,dy\,dx$$

114. If $f(x,\,y)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} C_{nm}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} D_{nm}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$ for

$(x,\,y)\in(0,\,a)\times(-b,\,b)$, then $C_{0m}=\tfrac{1}{ab}\int_0^a\int_{-b}^b f(x,\,y)\sin\tfrac{m\pi x}{a}\,dy\,dx$,

$$C_{nm}=\tfrac{2}{ab}\int_0^a\int_{-b}^b f(x,\,y)\sin\tfrac{m\pi x}{a}\cos\tfrac{n\pi y}{b}\,dy\,dx \text{ and } D_{nm}=\tfrac{2}{ab}\int_0^a\int_{-b}^b f(x,\,y)\sin\tfrac{m\pi x}{a}\sin\tfrac{n\pi y}{b}\,dy\,dx$$

115. If $f(x,\,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\cos\frac{(2n-1)\pi x}{2a}\cos\frac{(2m-1)\pi y}{2b}$ for $(x,\,y)\in(0,\,a)\times(0,\,b)$, then

$$C_{nm}=\tfrac{4}{ab}\int_0^a\int_0^b f(x,\,y)\cos\tfrac{(2n-1)\pi x}{2a}\cos\tfrac{(2m-1)\pi y}{2b}\,dy\,dx$$

116. If $f(x,\,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\sin\frac{n\pi x}{a}\cos\frac{(2m-1)\pi y}{2b}$ for $(x,\,y)\in(0,\,a)\times(0,\,b)$, then

$$C_{nm}=\tfrac{4}{ab}\int_0^a\int_0^b f(x,\,y)\sin\tfrac{n\pi x}{a}\cos\tfrac{(2m-1)\pi y}{2b}\,dy\,dx$$

117. If $f(x,\,y)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} C_{nm}\cos\frac{n\pi x}{a}\cos\frac{(2m-1)\pi y}{2b}$ for $(x,\,y)\in(0,\,a)\times(0,\,b)$, then

$$C_{0m}=\tfrac{2}{ab}\int_0^a\int_0^b f(x,\,y)\cos\tfrac{(2m-1)\pi y}{2b}\,dy\,dx \text{ and } C_{nm}=\tfrac{4}{ab}\int_0^a\int_0^b f(x,\,y)\cos\tfrac{n\pi x}{a}\cos\tfrac{(2m-1)\pi y}{2b}\,dy\,dx$$

118. If $f(x,\, y) = \displaystyle\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} \sin \frac{n\pi x}{a} \sin \frac{(2m-1)\pi y}{2b}$ for $(x,\, y) \in (0,\, a) \times (0,\, b)$, then

$$C_{nm} = \tfrac{4}{ab} \int_0^a \int_0^b f(x,\, y) \sin \tfrac{n\pi x}{a} \sin \tfrac{(2m-1)\pi y}{2b}\, dy\, dx$$

119. If $f(x,\, y) = \displaystyle\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} C_{nm} \cos \frac{n\pi x}{a} \sin \frac{(2m-1)\pi y}{2b}$ for $(x,\, y) \in (0,\, a) \times (0,\, b)$, then

$$C_{0m} = \tfrac{2}{ab} \int_0^a \int_0^b f(x,\, y) \sin \tfrac{(2m-1)\pi y}{2b}\, dy\, dx \text{ and } C_{nm} = \tfrac{4}{ab} \int_0^a \int_0^b f(x,\, y) \cos \tfrac{n\pi x}{a} \sin \tfrac{(2m-1)\pi y}{2b}\, dy\, dx$$

120. If $f(x,\, y) = \displaystyle\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} \sin \frac{(2n-1)\pi x}{a} \sin \frac{(2m-1)\pi y}{2b}$ for $(x,\, y) \in (0,\, a) \times (0,\, b)$, then

$$C_{nm} = \tfrac{4}{ab} \int_0^a \int_0^b f(x,\, y) \sin \tfrac{(2n-1)\pi x}{a} \sin \tfrac{(2m-1)\pi y}{2b}\, dy\, dx$$

Generalized Double Fourier Series

Suppose $0 < \alpha_1 < \alpha_2 < \cdots$ are zeros of $J_z(x)$.

121. If $f(r,\, \theta) = \displaystyle\sum_{k=0}^{\infty}\sum_{m=1}^{\infty} A_{mk} J_z(\frac{\alpha_m r}{a}) \cos \frac{k\pi\theta}{b}$ for $(r,\, \theta) \in (0,\, a) \times (0,\, b)$, then

$$A_{m0} = \tfrac{2}{a^2 b J_{z+1}^2(\alpha_m)} \int_0^a \int_0^b f(r,\, \theta) J_z(\tfrac{\alpha_m r}{a}) r\, d\theta\, dr \text{ and}$$

$$A_{mk} = \tfrac{4}{a^2 b J_{z+1}^2(\alpha_m)} \int_0^a \int_0^b f(r,\, \theta) \cos \tfrac{k\pi\theta}{b} J_z(\tfrac{\alpha_m r}{L}) r\, d\theta\, dr$$

122. If $f(r,\, \theta) = \displaystyle\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} B_{mk} J_z(\frac{\alpha_m r}{a}) \sin \frac{k\pi\theta}{b}$ for $(r,\, \theta) \in (0,\, a) \times (0,\, b)$, then

$$B_{mk} = \tfrac{4}{a^2 b J_{z+1}^2(\alpha_m)} \int_0^a \int_0^b f(r,\, \theta) \sin \tfrac{k\pi\theta}{b} J_z(\tfrac{\alpha_m r}{b}) r\, d\theta\, dr$$

123. If $f(r,\, \theta) = \displaystyle\sum_{k=0}^{\infty}\sum_{m=1}^{\infty} A_{mk} J_z(\frac{\alpha_m r}{a}) \cos \frac{k\pi\theta}{b} + \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} B_{mk} J_z(\frac{\alpha_m r}{a}) \sin \frac{k\pi\theta}{b}$ for

$(r,\, \theta) \in (0,\, a) \times (-b,\, b)$, then $A_{m0} = \tfrac{1}{a^2 b J_{z+1}^2(\alpha_m)} \displaystyle\int_0^a \int_{-b}^b f(r,\, \theta) J_z(\tfrac{\alpha_m r}{a}) r\, d\theta\, dr$,

$$A_{mk} = \tfrac{2}{a^2 b J_{z+1}^2(\alpha_m)} \int_0^a \int_{-b}^b f(r,\, \theta) \cos \tfrac{k\pi\theta}{b} J_z(\tfrac{\alpha_m r}{a}) r\, d\theta\, dr \text{ and}$$

$$B_{mk} = \tfrac{2}{a^2 b J_{z+1}^2(\alpha_m)} \int_0^a \int_{-b}^b f(r,\, \theta) \sin \tfrac{k\pi\theta}{b} J_z(\tfrac{\alpha_m r}{a}) r\, d\theta\, dr$$

124*. If $f(\theta,\, \phi) = \displaystyle\sum_{k=0}^{\infty}\sum_{n=m}^{\infty} A_{nk} P_n^m(\cos\phi) \cos \frac{k\pi\theta}{b}$ for $(\theta,\, \phi) \in (0,\, b) \times (0,\, \pi)$, then

$$A_{n0} = \tfrac{2n+1}{2b} \tfrac{(n-m)!}{(n+m)!} \int_0^\pi \int_0^b f(\theta,\, \phi) P_n^m(\cos\phi) \sin\phi\, d\theta\, d\phi \text{ and}$$

$$A_{nk} = \tfrac{2n+1}{b} \tfrac{(n-m)!}{(n+m)!} \int_0^\pi \int_0^b f(\theta,\, \phi) \cos \tfrac{k\pi\theta}{b} P_n^m(\cos\phi) \sin\phi\, d\theta\, d\phi$$

* It also holds if $k = m$. For $m = 0$, $P_n^m(s) = P_n(s)$ and $\frac{(n-m)!}{(n+m)!} = 1$.

125*. If $f(\theta,\,\phi)=\sum_{k=0}^{\infty}\sum_{n=m}^{\infty}A_{nk}P_n^m(\cos\phi)\sin\frac{k\pi\theta}{b}$ for $(\theta,\,\phi)\in(0,\,b)\times(0,\,\pi)$, then

$$B_{nk}=\tfrac{2n+1}{b}\tfrac{(n-m)!}{(n+m)!}\int_0^{\pi}\int_0^{b}f(\theta,\,\phi)\sin\tfrac{k\pi\theta}{b}P_n^m(\cos\phi)\sin\phi\,d\theta\,d\phi$$

126*. If $f(\theta,\,\phi)=\sum_{k=0}^{\infty}\sum_{n=m}^{\infty}A_{nk}P_n^m(\cos\phi)\cos\frac{k\pi\theta}{b}+\sum_{k=1}^{\infty}\sum_{n=m}^{\infty}B_{nk}P_n^m(\cos\phi)\sin\frac{k\pi\theta}{b}$ for

$(\theta,\,\phi)\in(-b,\,b)\times(0,\,\pi)$, then $A_{n0}=\tfrac{2n+1}{4b}\tfrac{(n-m)!}{(n+m)!}\int_0^{\pi}\int_{-b}^{b}f(\theta,\,\phi)P_n^m(\cos\phi)\sin\phi\,d\theta\,d\phi\,,$

$A_{nk}=\tfrac{2n+1}{2b}\tfrac{(n-m)!}{(n+m)!}\int_0^{\pi}\int_{-b}^{b}f(\theta,\,\phi)\cos\tfrac{k\pi\theta}{b}P_n^m(\cos\phi)\sin\phi\,d\theta\,d\phi$ and

$$B_{nk}=\tfrac{2n+1}{2b}\tfrac{(n-m)!}{(n+m)!}\int_0^{\pi}\int_{-b}^{b}f(\theta,\,\phi)\sin\tfrac{k\pi\theta}{b}P_n^m(\cos\phi)\sin\phi\,d\theta\,d\phi$$

* It also holds if $k=m$. For $m=0$, $P_n^m(s)=P_n(s)$ and $\frac{(n-m)!}{(n+m)!}=1$.

Generalized Triple Fourier Series

Suppose $0<\alpha_1<\alpha_2<\cdots$ are zeros of $J_z(x)$.

127**. If $f(\rho,\,\theta,\,\phi)=\sum_{q=0}^{\infty}\sum_{k=l}^{\infty}\sum_{m=1}^{\infty}A_{mkq}J_z(\frac{\alpha_m\rho}{a})P_k^l(\cos\phi)\cos\frac{q\pi\theta}{b}$ for $(\rho,\,\theta,\,\phi)\in(0,\,a)\times(0,\,b)\times(0,\,\pi)$,

then $A_{mk0}=\frac{2k+1}{a^2bJ_{z+1}^2(\alpha_m)}\frac{(k-l)!}{(k+l)!}\int_0^a\int_0^{\pi}\int_0^b f(\rho,\,\theta,\,\phi)J_z(\frac{\alpha_m\rho}{a})\rho P_k^l(\cos\phi)\sin\phi\,d\theta\,d\phi\,d\rho$ and

$$A_{mkq}=\tfrac{2(2k+1)}{a^2bJ_{z+1}^2(\alpha_m)}\tfrac{(k-l)!}{(k+l)!}\int_0^a\int_0^{\pi}\int_0^b f(\rho,\,\theta,\,\phi)J_z(\tfrac{\alpha_m\rho}{a})\rho P_k^l(\cos\phi)\sin\phi\cos\tfrac{q\pi\theta}{b}\,d\theta\,d\phi\,d\rho$$

128**. If $f(\rho,\,\theta,\,\phi)=\sum_{q=1}^{\infty}\sum_{k=l}^{\infty}\sum_{m=1}^{\infty}B_{mkq}J_z(\frac{\alpha_m\rho}{a})P_k^l(\cos\phi)\sin\frac{q\pi\theta}{b}$ for $(\rho,\,\theta,\,\phi)\in(0,\,a)\times(0,\,b)\times(0,\,\pi)$,

then $B_{mkq}=\frac{2(2k+1)}{a^2bJ_{z+1}^2(\alpha_m)}\frac{(k-l)!}{(k+l)!}\int_0^a\int_0^{\pi}\int_0^b f(\rho,\,\theta,\,\phi)J_z(\frac{\alpha_m\rho}{a})\rho P_k^l(\cos\phi)\sin\phi\sin\frac{q\pi\theta}{b}\,d\theta\,d\phi\,d\rho$

129**. If $f(\rho,\,\theta,\,\phi)=\sum_{q=0}^{\infty}\sum_{k=l}^{\infty}\sum_{m=1}^{\infty}A_{mkq}J_z(\frac{\alpha_m\rho}{a})P_k^l(\cos\phi)\cos\frac{q\pi\theta}{b}+$

$$\sum_{q=1}^{\infty}\sum_{k=l}^{\infty}\sum_{m=1}^{\infty}B_{mkq}J_z(\frac{\alpha_m\rho}{a})P_k^l(\cos\phi)\sin\frac{q\pi\theta}{b}$$

for $(\rho,\,\theta,\,\phi)\in(0,\,a)\times(-b,\,b)\times(0,\,\pi)$, then

$A_{mk0}=\frac{2k+1}{2a^2bJ_{z+1}^2(\alpha_m)}\frac{(k-l)!}{(k+l)!}\int_0^a\int_0^{\pi}\int_{-b}^b f(\rho,\,\theta,\,\phi)J_z(\frac{\alpha_m\rho}{b})\rho P_k^l(\cos\phi)\sin\phi\,d\theta\,d\phi\,d\rho\,,$

$A_{mkq}=\frac{2k+1}{a^2bJ_{z+1}^2(\alpha_m)}\frac{(k-l)!}{(k+l)!}\int_0^a\int_0^{\pi}\int_{-b}^b f(\rho,\,\theta,\,\phi)J_z(\frac{\alpha_m\rho}{a})\rho P_k^l(\cos\phi)\sin\phi\cos\frac{q\pi\theta}{b}\,d\theta\,d\phi\,d\rho$ and

$$B_{mkq}=\tfrac{2k+1}{a^2bJ_{z+1}^2(\alpha_m)}\tfrac{(k-l)!}{(k+l)!}\int_0^a\int_0^{\pi}\int_{-b}^b f(\rho,\,\theta,\,\phi)J_z(\tfrac{\alpha_m\rho}{a})\rho P_k^l(\cos\phi)\sin\phi\sin\tfrac{q\pi\theta}{b}\,d\theta\,d\phi\,d\rho$$

** For $l=0$, $P_k^l(s)=P_k(s)$ and $\frac{(k-l)!}{(k+l)!}=1$

Suppose $0 < \alpha_1 < \alpha_2 < \cdots$ are zeros of $J_z(x)$.

130*. If $f(\rho,\, \theta,\, \phi) = \displaystyle\sum_{q=0}^{\infty}\sum_{k=l}^{\infty}\sum_{m=1}^{\infty} A_{mkq}\, \rho^{-\frac{1}{2}} J_{z+\frac{1}{2}}(\frac{\alpha_m \rho}{a}) P_k^l(\cos\phi)\cos\frac{q\pi\theta}{a} +$

$$\sum_{q=1}^{\infty}\sum_{k=l}^{\infty}\sum_{m=1}^{\infty} B_{mkq}\, \rho^{-\frac{1}{2}} J_{z+\frac{1}{2}}(\frac{\alpha_m \rho}{a}) P_k^l(\cos\phi)\sin\frac{q\pi\theta}{b}$$

for $(\rho,\, \theta,\, \phi) \in (0,\, a) \times (-b,\, b) \times (0,\, \pi)$, then

$$A_{mk0} = \tfrac{2k+1}{2a^2 b J^2_{z+\frac{3}{2}}(\alpha_m)} \tfrac{(k-l)!}{(k+l)!} \int_0^a \int_0^\pi \int_{-b}^b f(\rho,\, \theta,\, \phi) J_{z+\frac{1}{2}}(\tfrac{\alpha_m \rho}{a}) \rho^{\frac{3}{2}} P_k^l(\cos\phi) \sin\phi \, d\theta \, d\phi \, d\rho \,,$$

$$A_{mkq} = \tfrac{2k+1}{a^2 b J^2_{z+\frac{3}{2}}(\alpha_m)} \tfrac{(k-l)!}{(k+l)!} \int_0^a \int_0^\pi \int_{-b}^b f(\rho,\, \theta,\, \phi) J_{z+\frac{1}{2}}(\tfrac{\alpha_m \rho}{a}) \rho^{\frac{3}{2}} P_k^l(\cos\phi) \sin\phi \cos\tfrac{q\pi\theta}{b} \, d\theta \, d\phi \, d\rho \text{ and}$$

$$B_{mkq} = \tfrac{2k+1}{a^2 b J^2_{z+\frac{3}{2}}(\alpha_m)} \tfrac{(k-l)!}{(k+l)!} \int_0^a \int_0^\pi \int_{-b}^b f(\rho,\, \theta,\, \phi) J_{z+\frac{1}{2}}(\tfrac{\alpha_m \rho}{a}) \rho^{\frac{3}{2}} P_k^l(\cos\phi) \sin\phi \sin\tfrac{q\pi\theta}{b} \, d\theta \, d\phi \, d\rho$$

* For $l = 0$, $P_k^l(s) = P_k(s)$ and $\frac{(k-l)!}{(k+l)!} = 1$

Chapter 0

Introduction (Not corresponding to the course textbook)

Definitions: 1. A partial differential equation (PDE) is an equation

$$F(u,\, u_x,\, u_y,\, \cdots,\, u_{xx},\, u_{xy},\, \cdots) = G(x,\, y,\, \cdots)$$

involving independent variables $x,\, y,\, \cdots,$ a function u of these variables and the partial derivatives $u_x,\, u_y,\, \cdots,\, u_{xx},\, u_{xy},\, \cdots,$ of the function. Also, functions of independent variables may be used as coefficients for function u and its partial derivatives.

2. The order of a PDE is the order of the partial derivative of highest order appearing in the equation.

3. A function $u(x,\, y,\, \cdots)$ is called a solution of the PDE if the PDE becomes an identity in the independent variables when u and its partial derivatives are substituted in the PDE.

4. A PDE is called homogeneous if $G \equiv 0$, (no independent variable appears by itself).

5. A PDE is called linear if for all constants α and β and functions u and v we have

$$F(w,\, w_x,\, w_y,\, \cdots,\, w_{xx},\, w_{xy},\, \cdots) = \alpha F(u,\, u_x,\, u_y,\, \cdots,\, u_{xx},\, u_{xy},\, \cdots) + \beta F(v,\, v_x,\, v_y,\, \cdots,\, v_{xx},\, v_{xy},\, \cdots)$$

where $w = \alpha u + \beta v$.

Examples: 1. $\frac{\partial u}{\partial t} - u\frac{\partial u}{\partial x} = 0$ is a nonlinear homogeneous first-order PDE.

For nonlinearity, show $F(\alpha u + \alpha v) \neq \alpha F(u) + \beta F(v)$ for particular values of α, β and functions u and v, where $F(u) = \frac{\partial u}{\partial t} - u\frac{\partial u}{\partial x}$. *Classroom discussion!*

2. $u_{xx} + u_{yy} = 6x$ is a linear nonhomogeneous second-order PDE and $u(x,\, y) = x^3 + x^2 - y^2$ and $u(x,\, y) = x^3 + e^x \cos y$ are two solutions of it.

Classroom discussion!

3. $u_{xx} = \frac{1}{k} u_t$ is a homogeneous linear PDE of order 2.

Here $F(u) = u_{xx} - \frac{1}{k} u_t$. *Classroom discussion!*

Examples: 1. Find the solution $u(x, y)$ of $\frac{\partial u}{\partial x} - y \sin x = 0$.

For how to solve a separable ODE see the Review, Identities, Formulas and TheoremsHandout. *Classroom discussion!*

2. Find the solution $u(x, y)$ of $u_{xx} - u = 0$ which satisfies the auxiliary conditions; $u(0, y) = y + 6$ and $u_x(0, y) = y$.

For how to solve a 2nd order linear ODE with constant coefficients see the Review, Identities, Formulas and TheoremsHandout. *Classroom discussion!*

Exercises: 1. In example (1) above, have we found all of the solutions?
2. Find the solution $u(x,\, y,\, z)$ of $u_x - y \sin x = 0$.
3. Find the solution $u(x,\, y)$ of $u\, u_{xy} + u_x\, u_y = 0$. Hint: Notice that it is an exact ODE with respect to y derivative whose solution leads to a separable equation. For how to solve an exact ODE see the Review, Identities, Formulas and TheoremsHandout.

Sometimes we can find infinitely many solutions. For example consider $u_x + u_y = 0$. Functions $u_n(x,\, y) = (x-y)^n$, $n = 0,\, 1,\, \cdots$, satisfy the PDE, so perhaps their "infinite linear combination" $u(x,\, y) = \sum_{n=0}^{\infty} c_n u_n(x,\, y)$ will also be a solution of this PDE. Notice that if we take $c_n = \frac{1}{n!}$, then $u(x,\, y) = \sum_{n=0}^{\infty} \frac{(x-y)^n}{n!} = e^{x-y}$ which is a solution of our PDE.

The typical problem is to find a solution of a PDE which satisfies certain auxiliary conditions, for example;

$$u_{xx} = \frac{1}{k} u_t, \quad 0 < x < a,\ t > 0 \quad \text{(Heat Equation)}$$

$$\begin{array}{l} \text{Auxiliary} \\ \text{Conditions} \end{array} \left\{ \begin{array}{ll} \left. \begin{array}{l} u(0,\, t) = T_0, \ \ t > 0 \\ u(a,\, t) = T_1, \ \ t > 0 \end{array} \right\} & \text{Boundary Conditions} \\ u(x,\, 0) = f(x), \ \ 0 < x < a & \text{Initial Condition} \end{array} \right.$$

Our main solution technique will be the method of separation of variables, also called product method and Fourier's method.

Example. Solve $u_t - u\, u_x = 0$ by separation of variables.

Assume $u(x,\, t) = \phi(x) h(t)$, plug into the PDE and simplify to get $\phi'(x) = \frac{h'(t)}{h(t)}$. Since the left hand side (LHS) is a function of x and RHS is a function of t, this equality will hold only if they are equal to a constant, say λ. Solve $\phi'(x) = \lambda$ and $\frac{h'(t)}{h^2(t)} = \lambda$ and plug back in the function u. *Classroom discussion!*

Exercise. Solve $u_t = u_{xx}$ by the method of separation of variables. Hint: Consider the cases $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$.

Chapter 1

Fourier Series (Fourier Series and Integrals in the course textbook)

1.1 Periodic Functions and Fourier Series

Definition. A function f is said to be periodic with positive period p if

1. $f(x)$ has been defined for all x, and
2. $f(x+p) = f(x)$ for all x.

For a periodic function f with period p, it is easy to show that $f(x-np) = f(x) = f(x+np)$ for $n = 0,\ 1,\ \cdots$ and thus a periodic function, defined as above, has many periods! *Classroom discussion!*

Examples: 1. PUT GRAPH HERE!
2. PUT GRAPH HERE!
3. Functions $\sin x$ and $\cos x$ are 2π-periodic. Functions $\sin \frac{2\pi x}{p}$ and $\cos \frac{2\pi x}{p}$ have period p and the period of the function $\sin \frac{5\pi x}{3}$ is $\frac{2\pi}{\frac{2\pi}{3}} = \frac{6}{5}$.

Let f be a $2a$-periodic function (of period $2a$). In this chapter we want to find constants a_0, a_n, b_n, $n = 1,\ 2,\ \cdots$ such that

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{a} + b_n \sin \frac{n\pi x}{a} \right).$$

Orthogonatlity Relations

Assume m and n are nonnegative integers, unless stated otherwise.

$$\int_{-\pi}^{\pi} \sin nx\, dx = 0 \text{ for all } n$$

$$\int_{-\pi}^{\pi} \cos nx\, dx = \begin{cases} 0, & n \neq 0 \\ 2\pi, & n = 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx\, dx = 0 \text{ for all } n \text{ and } m$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx\, dx = \begin{cases} 0, & n \neq m \\ \pi, & n = m \end{cases}$$

$$\int_{-\pi}^{\pi} \cos nx \cos mx\, dx = \begin{cases} 0, & n \neq m \\ \pi, & n = m \neq 0 \\ 2\pi, & n = m = 0 \end{cases}$$

See Review, Identities, Formulas and Theorems. These are easy to check, (do it). We will need the following identities. See Review, Identities, Formulas and Theorems.

$$\begin{aligned}\sin a\cos b &= \tfrac{1}{2}(\sin(a+b)+\sin(a-b))\\ \cos a\cos b &= \tfrac{1}{2}(\cos(a+b)+\cos(a-b))\\ \sin a\sin b &= \tfrac{1}{2}(\cos(a-b)-\cos(a+b))\end{aligned}$$

We will use the first identity to prove the third orthogonality relation above.

$\displaystyle\int_{-\pi}^{\pi}\sin nx\cos mx\,dx = \int_{-\pi}^{\pi}(\tfrac{1}{2}\sin(n+m)x+\tfrac{1}{2}\sin(n-m)x)\,dx = 0$, using the first orthogonality relation.

This can be treated in a more general form.

Definitions: 1. Function $f(x)$ is an even function if $f(-x)=f(x)$ or equivalently its graph is symmetric about y-axis.
2. Function $f(x)$ is an odd function if $f(-x)=-f(x)$ or equivalently its graph is symmetric about the origin.

Then we have

$$\begin{aligned}&\int_{-a}^{a}(\text{even function})\,dx = 2\int_{0}^{a}(\text{even function})\,dx \text{ and}\\ &\int_{-a}^{a}(\text{odd function})\,dx = 0\end{aligned}$$

$\cos x$ is an even function, while $\sin x$ is an odd function. We can also think of even as "+" and odd as "-" in the following sense: even $\times$ odd $=$ odd, odd $\times$ odd $=$ even, even $\times$ even $=$ even, even $+$ even $=$ even and odd $+$ odd $=$ odd.

Each function can be written as the sum of an even function and an odd function.

$$f(x) = \underbrace{\frac{1}{2}(f(x)+f(-x))}_{\text{even}} + \underbrace{\frac{1}{2}(f(x)-f(-x))}_{\text{odd}}$$

Therefore since $\sin nx$ is odd and $cosmx$ is even, their product is odd and so $\int_{-\pi}^{\pi}\sin nx\cos mx\,dx = 0$.

Now suppose f is a 2π-periodic function and that $f(x) = a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$. By interchanging the order of integration and infinite sum, we can show the following.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx = \begin{cases} 0, & f \text{ odd}\\ \frac{1}{\pi}\int_0^{\pi}f(x)\,dx, & f \text{ even}\end{cases}$$

For any fixed integer value $m = 1,\, 2,\, \cdots,$

$$\begin{aligned}a_m &= \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos mx\,dx = \begin{cases} 0, & f \text{ odd}\\ \frac{2}{\pi}\int_0^{\pi}f(x)\cos mx\,dx, & f \text{ even}\end{cases}, \text{ and}\\ b_m &= \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin mx\,dx = \begin{cases} \frac{2}{\pi}\int_0^{\pi}f(x)\sin mx\,dx, & f \text{ odd}\\ 0, & f \text{ even}\end{cases}\end{aligned}$$

Classroom discussion!

Definition. Fourier series (F.S.) of 2π-periodic function $f(x)$ is defined as

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$ for $n = 1, 2, \cdots$.
Question: Does this series actually represent function $f(x)$?

Example. Find F.S. of $f(x) = |x|$ for $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$.

$f(x)$ is an even function with period 2π and the graph as shown. PUT THE GRAPH HERE.

So, b_n=0 and $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx == \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx = \cdots = \frac{\pi}{2}$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} x\cos nx\,dx = \cdots = \frac{2}{\pi n^2}(\cos n\pi) - 1 = \begin{cases} -\frac{4}{\pi n^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$, using integration-by-parts. Or, $a_{2n} = 0$ and $a_{2n-1} = -\frac{4}{(2n-1)^2}$ for $n = 1, 2, \cdots$. Thus

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos(2n-1)x\,.$$

Classroom discussion!

1.2 Arbitrary Period and Half-Range Expansions

Definition. Let f be a periodic function of period $2a$, then its F.S. is

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{a} + b_n \sin \frac{n\pi x}{a} \right)$$

where $a_0 = \frac{1}{2a} \int_{-a}^{a} f(x)\, dx$, $a_n = \frac{1}{a} \int_{-a}^{a} f(x) \cos \frac{n\pi x}{a}\, dx$ and $b_n = \frac{1}{a} \int_{-a}^{a} f(x) \sin \frac{n\pi x}{a}\, dx$ for $n = 1, 2, \cdots$.

Exercise. As we did for the 2π-periodic functions, derive above equations for a_0, a_n, b_n, $n = 1, 2, \cdots$.

What if f is defined only on a finite interval, say $(-a, a)$? For example, let's find the F.S. of $f(x) = \begin{cases} 1, & 0 < x < a \\ -1, & -a < x < 0 \end{cases}$. $f(x)$ is not defined on all of the $\mathbb{R}$ (real numbers), so to use the above formulas, we extend f to all of $\mathbb{R}$ in such a way that it will have period $2a$. Call this new function $\overline{f}$. F.S. of f on $(-a, a)$ is the F.S. of $\overline{f}$ on $(-a, a)$.

PUT GRAPH HERE

Note. More often than not we simply call this new function f again.

Since $\overline{f}$ is an odd function, $a_0 = a_n = 0$ and $b_n = \frac{2}{a} \int_0^a \overline{f}(x) \sin \frac{n\pi x}{a}\, dx = \cdots = \frac{-2}{n\pi}(\cos n\pi - 1) = \begin{cases} 0, & n \text{ even} \\ \frac{4}{n\pi}, & n \text{ odd} \end{cases}$ or $b_{2n} = 0$ and $b_{2n-1} = \frac{4}{(2n-1)\pi}$, $n = 1, 2, \cdots$. Thus $f(x) \sim \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \sin \frac{(2n-1)\pi x}{a}$.

Classroom discussion!

If function $f(x)$ is not defined on the interval $(-a, a)$, then we make either an odd or an even extension of f to all of $\mathbb{R}$.

Examples: Make even and odd extensions of the following functions to the entire real-number line.
1. $f(x) = x$, $0 < x < 1$ **2.** $f(x) = \sin x$, $0 < x < \pi$ **3.** $f(x) = (x-2)^2$, $2 < x < 3$

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These extensions are not unique, for example, for example 3 we could use the following extensions.

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Since in either case evaluation of a's and b's will only involve integration on $(0, a)$, we call the F.S. for these cases Half-Range Expansion.

Definitions: Let f be a function defined on $(0, a)$.

1. The odd extension of f to $(-a, a)$ is $f_o(x) = \begin{cases} f(x), & 0 < x < a \\ -f(-x), & -a < x < 0 \end{cases}$.

The F. sine series of f is the F. series of f_o.

2. The even extension of f to $(-a, a)$ is $f_e(x) = \begin{cases} f(x), & 0 < x < a \\ f(-x), & -a < x < 0 \end{cases}$.

The F. cosine series of f is the F. series of f_e.

Examples: Consider the function f, in example 2 above.

1. Find the F. cosine series of f.
Using f_e, the even extension of f to all of $\mathbb{R}$, we have

$$b_n = 0$$
$$a_0 = \tfrac{2}{2\pi}\int_0^\pi \sin x\, dx = \cdots = \tfrac{2}{\pi}$$
$$a_n = \tfrac{2}{\pi}\int_0^\pi \sin x \cos nx\, dx = \cdots = \begin{cases} 0, & n = 1 \\ 0, & n > 1 \text{ and } n \text{ odd} \\ \frac{-4}{\pi(n^2-1)}, & n > 1 \text{ and } n \text{ even} \end{cases} \implies$$
$$a_{2n} = \tfrac{-4}{\pi(4n^2-1)},\ a_{2n-1} = 0,\ n = 1,\, 2,\, \cdots$$
$$f(x) \sim \frac{2}{\pi} - \sum_{n=1}^{\infty} \frac{4}{(4n^2-1)\pi} \cos 2nx$$

Classroom discussion!

2. Find the F. sine series of f.

The odd extension of f to all of $\mathbb{R}$ is $\sin x$ itself. So, F. sine series of f should be $f(x) \sim \sin x$.

Question. At $x = \frac{\pi}{2}$, $\sin x = 1$ and $\sum_{n=1}^{\infty} \frac{4}{(4n^2-1)\pi} \cos 2nx = \sum_{n=1}^{\infty} \frac{4(-1)^n}{(4n^2-1)\pi}$. Is $\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{(4n^2-1)} = 1$?

Exercise. Find the F. sine series of $f(x) = \begin{cases} \frac{2x}{a}, & 0 < x \le \frac{a}{2} \\ 2 - \frac{2x}{a}, & \frac{a}{2} \le x < a \end{cases}$.

Answer: $f(x) \sim \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi}{2}}{n^2} \sin\frac{n\pi x}{a} = \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin\frac{(2n-1)\pi x}{a}$

Definitions: Let function f be defined on $(0, a)$.

1. The Fourier sine series of f is $\sum_{n=1}^{\infty} b_n \sin \frac{n\,\pi\,x}{a}$ where $b_n = \frac{2}{a}\int_0^a f(x) \sin \frac{n\,\pi\,x}{a}\,dx$.

2. The Fourier cosine series of f is $a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\,\pi\,x}{a}$ where $a_0 = \frac{1}{a}\int_0^a f(x)\,dx$ and $a_n = \frac{2}{a}\int_0^a f(x)\cos \frac{n\,\pi\,x}{a}\,dx$.

1.3 Convergence of Fourier Series

Definition. Let $f(x)$ be a function and $x_0 \in \mathbb{R}$. We say that $\lim_{x\to x_0} f(x)$ exists if

1. left limit exists $\iff f(x_0^-) = \lim_{x\to x_0^-} f(x) = \lim_{h\to 0^-} f(x_0+h) = \lim_{\substack{h\to 0\\ h<0}} f(x_0+h)$ exits,
2. right limit exists $\iff f(x_0^+) = \lim_{x\to x_0^+} f(x) = \lim_{h\to 0^+} f(x_0+h) = \lim_{\substack{h\to 0\\ h>0}} f(x_0+h)$ exits, and
3. the above two limits are equal $\iff f(x_0^-) = f(x_0^+)$.

Then $\lim_{x\to x_0} f(x) = f(x_0^-) = f(x_0^+)$.

Definition. Function $f(x)$ is continuous at x_0 if

$$\left.\begin{array}{l} 1.\ f(x_0) \text{ exits,} \\ 2.\ \lim_{x\to x_0} f(x) \text{ exits, and} \\ 3.\ f(x_0) = \lim_{x\to x_0} f(x). \end{array}\right\} \iff f(x_0^-) = f(x_0^+) = f(x_0)$$

Definition. A function is continuous (everywhere) if it is continuous at each point.

Types of Discontinuity at x_0- **1.** Removable discontinuity; $f(x_0^-) = f(x_0^+) \neq f(x_0)$, ($f(x_0$ may not be defined.)
2. Jump discontinuity; $f(x_0^-) \neq f(x_0^+)$, but both exist. **3.** "Bad" discontinuity; $f(x_0^-)$, $f(x_0^+)$ or both fail to exist.

Examples: 1. **2.**

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3. **4.** **5.**

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In the case of removable discontinuity at x_0, if we redefine our function at x_0 to be $f(x_0) = \lim_{x\to x_0} f(x)$, then this new function is continuous at x_0, hence the terminology "removable discontinuity".

Definition. A function is piecewise continuous on a finite interval (a, b), if it is bounded and continuous on (a, b), except possibly for a finite number of jumps and removable discontinuities.

Definition. A function is piecewise continuous if it is piecewise continuous on every finite interval.

Remark. Another name for piecewise continuous is piecewise continuous.

Examples: Addition to the parts of the last example.
1. Function f is not piecewise continuous on $(0, 2x_0)$, so it is not piecewise continuous.
4. Function f is piecewise continuous on each finite interval, so it is not piecewise continuous.

If a function defined on a finite interval is piecewise continuous, then its periodic (odd or even) extension is also piecewise continuous, for example consider

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Definition. A function f is piecewise smooth if
1. f is piecewise continuous,
2. $f'(x)$ exists for every x, except perhaps at a finite number of points, and
3. $f'(x)$ is piecewise continuous.

Examples: 1. $f(x) = x^{\frac{1}{3}}$, $-1 < x < 1$.
f is continuous on , f' is not continuous on and therefore f is not piecewise smooth. *Classroom discussion!*

2. $f(x) = |x|$, $-1 < x < 1$.
f is continuous, although $f'(0)$ does not exist f' is piecewise continuous and therefore f is piecewise smooth. *Classroom discussion!*

Theorem (Convergence Theorem, Function Hypotheses)**.** If $f(x)$ is piecewise smooth and periodic with period $2a$, then at each point x the F.S. corresponding to f converges and

$$a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{a} + b_n \sin \frac{n\pi x}{a} \right) = \frac{1}{2} \left(f(x^-) + f(x^+) \right).$$

Remark. If f is continuous at x, then $\frac{1}{2}(f(x^-) + f(x^+)) = f(x)$.

Example. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{4(-1)^n}{4n^2 - 1} = 2 - \pi$.
Apply the above theorem to the F. cosine series of $f(x) = \sin x$, $0 < x < \pi$ and then plug in $x = \frac{\pi}{2}$. *Classroom discussion!*

Example. The graph of periodic function f is shown below. Draw the graph of its F. series.

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Classroom discussion!

Remark. The above examples show the power of the Convergence Theorem.

Exercises: 1. Use the F.S. of $f(x) = |\sin x|$ to show that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} = \frac{1}{2} - \frac{\pi}{4}$.

2. Use the F. cosine series of $f(x) = x^2$, $0 < x < \pi$, to show that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

3. Use the F. cosine series of $f(x) = x^4$, $0 < x < \pi$, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ to show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

4. Consider the $2a$-periodic function f with $f(x) = \begin{cases} -1, & -a \le x < 0 \\ 3, & 0 \le x < a \end{cases}$. Find its F. series graphically. (Do not compute its F.S. coefficients.)

It is also useful to state convergence theorems for the F. series when f is defined on $(-a, a)$ and both F. sine and cosine series when f is defined on $(0, \mathrm{a})$. We need to find the conditions for which the desired extension of f meets the hypotheses of the Convergence Theorem. We must also pay special attention to the endpoints.

Exercises: Fill in the blank.
1. Let $f(x)$ be a function defined on $(-a, a)$. If f is   , then F.S. corresponding to f converges and

$$a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{a} + b_n \sin \frac{n\pi x}{a} \right) = \frac{1}{2} \left(f(x^-) + f(x^+) \right)$$

for $-a < x < a$. At $x = \pm a$, the F.S. converges to .
2. Let f be a function defined on $(0, a)$. If f is , then F. sine series corresponding to f converges and

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a} = \frac{1}{2}\left(f(x^-) + f(x^+)\right)$$

for $0 < x < a$. At $x = 0$ and $x = a$, the F. sine series converges to .
3. Let $f(x)$ be a function defined on $(0, a)$. If f is , then F. cosine series corresponding to f converges and

$$a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{a} = \frac{1}{2}\left(f(x^-) + f(x^+)\right)$$

for $0 < x < a$. At $x = 0$, the F. cosine series converges to . At $x = a$, the F. cosine series converges to .

1.4 Uniform Convergence

Definitions: Consider functions $f_n(x)$, $n = 1, 2, \cdots$ defined on the interval I.

1. We say that $\sum_{n=1}^{\infty} f_n(x)$ converges to $f(x)$ **pointwise** in the interval I if at each point x in I, $\lim_{N\to\infty} \left|\sum_{n=1}^{N} f_n(x) - f(x)\right| = 0$.

$\sum_{n=1}^{N} f_n(x)$ is called the Nth partial sum of $\sum_{n=1}^{\infty} f_n(x)$ and is denoted by $S_N(x)$; sum of the 1st N term.

2. We say that $\sum_{n=1}^{\infty} f_n(x)$ converges to $f(x)$ **uniformly** in the interval I if $\lim_{N\to\infty} \operatorname{Max}_{x\in I} |S_N(x) - f(x)| = 0$.

Note. This maximum might not exist, in that case we use supremum; the least number greater than $|S_N(x) - f(x)|$ for every x, in place of it.

Lemma. If $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to, say, $f(x)$ and if $f_n(x)$ are continuous functions, then $f(x)$ is also continuous.

Examples: Examine convergence of F.S. of following functions graphically.

1. $f(x) = \begin{cases} 1, & 0 < x < \pi \\ -1, & -\pi < x < 0 \end{cases}$

$$f(x) \sim \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \sin(2n-1)x$$

2. $g(x) = |x|$, $-\pi < x < \pi$

$$g(x) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)x$$

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F.S. does not converge uniformly to $f(x)$.

F.S. converges uniformly to $g(x)$.

Theorem 1. (Convergence Theorem, F. Coefficients Hypotheses)
Consider the series $a_0 + \sum_{n=1}^{\infty}\left(a_n \cos \frac{n\pi x}{a} + b_n \sin \frac{n\pi x}{a}\right)$. If $\sum_{n=1}^{\infty} (|a_n| + |b_n|)$ converges then this series converges uniformly (and hence to a continuous function) and if it is the F.S. of the function $f(x)$, it converges uniformly to $f(x)$.

Example. Use the above theorem to show that the F.S. of $g(x)$ in the last example converges uniformly to $g(x)$.

Classroom discussion!

Remark. Notice that for the function f in the last example, $\sum_{n=1}^{\infty}(|a_n|+|b_n|) = \sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}$ does not converge and hence the above uniform convergence theorem does not hold.

The following theorems state conditions under which F.S. of a function converges uniformly. These conditions do not involve the F.S. itself.

Theorem 2. If function $f(x)$ is periodic, continuous, and has a piecewise continuous derivative, then F.S. corresponding to f converges uniformly to f on the entire real axis.

Theorem 3. Let $f(x)$ be a function defined on $(-a,\, a)$ such that

1. It is continuous, bounded (bdd $\leftrightarrow |f(x)| < M$ for all x and some $M > 0$) and has piecewise continuous derivative, and
2. $f((-a)^+) = f(a^-)$.

Then the F.S. of f converges uniformly to f on the interval $(-a,\, a)$. (F.S. converges to $f((-a)^+) = f(a^-)$ at $x = \pm a$.)

Theorem 4. Let $f(x)$ be a function defined on $(0,\, a)$ such that

1. It is continuous, bounded, and has piecewise continuous derivative, and
2. $f(0^+) = f(a^-) = 0$

Then the F. sine series of f converges uniformly to f in the interval $(0,\, a)$. (F. sine series converges to zero at $x = 0$ and $x = a$.)

Theorem 5. Let $f(x)$ be a function defined on $(0,\, a)$ such that it is continuous, bounded, and has piecewise continuous derivative. Then the F. cosine series of f converges uniformly to f in the interval $(0,\, a)$. (F. cosine series converges to $f(0^+)$ at $x = 0$ and to $f(a^-)$ at $x = a$.)

Exercises: 1. Show that $f(x) = |x|$, $-\pi < x < \pi$, satisfies the hypothesis of theorem 3.
2. Show that $f(x) = \sin x$, $0 < x < \pi$, satisfies the hypothesis of both theorems 4 and 5. What are its F. sine and cosine series? Are they equal?

1.5 Operations on Fourier Series

Let $f(x)$ be a $2a$-periodic function and $a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{a} + b_n \sin\frac{n\pi x}{a}\right)$ its Fourier series.

Theorem 1. The F.S. of function $cf(x)$ has coefficients ca_0, ca_n and cb_n, where c is any constant.

Theorem 2. The Fourier coefficients of the sum $f(x)+g(x)$ are the sums of the corresponding coefficients of F.S. of $f(x)$ and $g(x)$.

Exercise. Prove theorems 1 and 2, by direct computation of F. coefficients of $cf(x)$ and $f(x)+g(x)$.

Theorem 3. If $f(x)$ is a $2a$-periodic piecewise continuous function, then F.S. of f may be integrated term by term. That is,

$$\int_c^d f(x)\,dx = \int_c^d a_0\,dx + \sum_{n=1}^{\infty} \int_c^d \left(a_n \cos \tfrac{n\pi x}{a} + b_n \sin \tfrac{n\pi x}{a}\right)\,dx.$$

Theorem 4. If $f(x)$ is a $2a$-periodic piecewise continuous function and function $g(x)$ is also piecewise continuous on (c, d), then

$$\int_c^d f(x)g(x)\,dx = \int_c^d a_0 g(x)\,dx + \sum_{n=1}^{\infty} \int_c^d \left(a_n \cos \tfrac{n\pi x}{a} + b_n \sin \tfrac{n\pi x}{a}\right) g(x)\,dx.$$

Remarks: 1. The hypotheses in theorems 3 and 4 are weaker than that of the Convergence Theorem. We are not requiring convergence of the F.S. to the function!
2. The term-by-term integration of a F.S. may not result in another F. series!

Theorem 5. (Uniqueness Theorem) If $f(x)$ is periodic and piecewise continuous, then its F.S. is unique.

Remark. If $f(x)$, $0 < x < a$ is piecewise continuous, then its F. sine and cosine series are unique.

Example. Consider $f(x) = |x|$, $-\pi < x < \pi$. Its F.S. is $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x$ (equality is due to convergence theorem).

1. Find F.S. of $h(x) = \frac{\pi^2}{8} - \frac{\pi}{4}f(x)$, $-\pi < x < \pi$.
2. Evaluate $\displaystyle\int_0^x h(t)\,dt$, by use of theorem 3.
3. Evaluate $\displaystyle\int_0^x h(t)\,dt$, $0 < x < \pi$ directly.
4. Find F. sine series of $g(x) = \frac{\pi}{8}x(\pi - x)$, $0 < x < \pi$.
5. Show that $\frac{2}{\pi}\int_0^\pi g(x)\sin(2n-1)x\,dx = \frac{1}{(2n-1)^3}$, $n = 1, 2, \cdots$.
6. Show that $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}$.

Classroom discussion!

In the following exercise, we will see some issues that arise from term-by-term integration of a F.S., including not being a F. series.

Exercise. The F. sines series of $f(x) = x$, $0 < x < \pi$, is $x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$, $0 \leq x < \pi$. (Equality is due to the convergence theorem.)

1. Show $x^2 = -\sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$, $0 < x < \pi$.

2. Find the value of $-\sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}$ by noticing that it is the constant in the F. cosine series of $g(x) = x^2$, $0 < x < \pi$.

3. Using earlier parts, show that the F. cosine series of $g(x) = x^2$, $0 < x < \pi$, is $g(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx$, $0 \le x \le \pi$, and use it to show $x^3 = \pi^2 x + \sum_{n=1}^{\infty}\frac{12(-1)^n}{n^3}\sin nx$, $0 < x < \pi$. (Note: The series $\pi^2 x + \sum_{n=1}^{\infty}\frac{12(-1)^n}{n^3}\sin nx$ is not a F. series!)

4. Use earlier parts to find the F. sine series of $h(x) = x^3$, $0 < x < \pi$.

The following exercise demonstrates a very interesting property of F. coefficients.

Exercises: 1. Let f be an odd periodic piecewise continuous function with period $2a$. Show that $\frac{1}{a}\int_{-a}^{a} f^2(x)\,dx = \sum_{n=1}^{\infty} b_n^2$, where b_n's are the coefficients of F. sine series of f.

Remark. This is a form of Parseval's equality.

2. Let f be an odd periodic piecewise continuous function with period $2a$. Show that $\lim_{n\to\infty} b_n = 0$, where b_n's are the coefficients of F. sine series of f. Hint: Divergence Test - If $\sum_{n=1}^{\infty} c_n$ converges, then $\lim_{n\to\infty} c_n = 0$. (If $\lim_{n\to\infty} c_n \neq 0$, then $\sum_{n=1}^{\infty} c_n$ diverges.)

We will use the following result when we apply comparison theorem to series involving F. coefficients.

Lemma. If sequence $\{b_n\}_{n=1}^{\infty}$ converges then it is bounded. That is, there exists a number $M > 0$ such that $|b_n| < M$ for every n.

The following example demonstrates that term-by-term differentiation of a F. series is not always possible.

Example. The F. sines series of $f(x) = x$, $0 < x < a$, is $x = \sum_{n=1}^{\infty}\frac{2a}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{a}$, $0 \le x < a$. (Equality is due to the convergence theorem.) Show that the F. series of $f'(x)$ is not the term-by-term differentiated F. series of $f(x)$.

Classroom discussion!

Theorem 6. (Term-by-Term Differentiation Theorem, Function Hypotheses) If $f(x)$ is 2a-periodic, continuous, and piecewise smooth, then the term-by-term differentiated F.S. of $f(x)$ converges to $f'(x)$ at every point x where $f''(x)$ exists.

$$f'(x) = \sum_{n=1}^{\infty}\left(-\tfrac{n\pi}{a}a_n\sin\tfrac{n\pi x}{a} + \tfrac{n\pi}{a}b_n\cos\tfrac{n\pi x}{a}\right), \text{ where } f''(x) \text{ exists.}$$

Example. Find the derivative of $\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x$ using the fact that it is the F.S. of $f(x)=|x|$, $-\pi<x<\pi$.

Classroom discussion!

Theorem 7. (Term-by-Term Differentiation Theorem, F. Coefficients Hypotheses) If $f(x)$ is periodic with F. coefficients a_0, a_n and b_n, and if the series $\sum_{n=1}^{\infty}\left(|n^k a_n|+|n^k b_n|\right)$ converges for an integer $k\geq 1$, then f has continuous derivatives $f', \cdots, f^{(k)}$ whose F.S. are the corresponding term-by-term differentiated series of f.

Remark. Suppose the above theorem holds. Then, the F. coefficient of $f^{(k)}(x)$ are $\pm(\frac{n\pi}{a})^k a_n$ and $\pm(\frac{n\pi}{a})^k b_n$. Also, $\sum_{n=1}^{\infty}\left(|\pm(\frac{n\pi}{a})^k a_n|+|\pm(\frac{n\pi}{a})^k a_n|\right)=(\frac{\pi}{a})^k\sum_{n=1}^{\infty}\left(|n^k a_n|+|n^k b_n|\right)$ converges. This means that not only the F.S. of $f^{(k)}(x)$ is obtained by k term-by-term differentiation of the F.S. of $f(x)$, but also that the F.S. of $f^{(k)}(x)$ is **equal** to the function $f^{(k)}(x)$ itself, due to theorem 1 in section 1.4.

Example. Given $u(x)\sim\sum_{n=1}^{\infty}Me^{-n^2 t}\sin nx$, where M and t are fixed positive constants. Find the F. series of $\frac{d^2u}{dx^2}$. Discuss the convergence of the F. series of $\frac{d^2u}{dx^2}$.

Classroom discussion!

The theorems in this section for interchanging the order of summation and integration or differentiation are for F. series only and are based on the following general results.

Theorem 8. (Interchanging Integral and Summation) See Review, Identities, Formulas and Theorems.

Theorem 9. (Interchanging Differentiation and Summation) See Review, Identities, Formulas and Theorems.

A convenient way to show uniform convergence is through Weierstrass M Test.

Theorem 10. (Weierstrass M Test) See Review, Identities, Formulas and Theorems.

Note. Theorems 8-10 are not in the textbook, but are stated in Review, Identities, Formulas and Theorems.

In the problem below, we need to find the derivative of a series which is not a F.S. with the respect to the variable we must differentiate. In this case, we will use the above theorem for interchanging differentiation and summation.

Example. (Mathematical Justification) Let f be an odd, periodic, piecewise smooth function with F. sine series coefficients b_n, $n = 1, 2, \cdots$. Show that the function defined by $u(x, t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin nx$ satisfies the following.

a. $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $0 < x < \pi$, $t > 0$

b. $u(0, t) = u(\pi, t) = 0$, $t > 0$

c. $u(x, 0) = \frac{1}{2}(f(x^-) + f(x^+))$, $0 < x < \pi$

Classroom discussion!

Chapter 2

The Heat Equation

2.1 Derivation and Boundary Conditions

We want to obtain the equation governing the flow of the heat in a thermally conducting rod whose solution gives the temperature at any given position on the rod at any given time.

Assume the rod has a uniform cross section and that the temperature does not vary form point to point in a cross section. Therefore the temperature in the rod will only depend on position x and time t.

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We will make use of the following.

1. The law of Conservation of Energy - The amount of heat which enters a region plus what is generated inside is equal to the amount of the heat which leaves plus the amount stored; Heat in + Heat generated = Heat out + Heat stored. This is equally valid in terms of rates of heat per unit of time instead of amounts of heat.

2. The rate of heat stored in a body is proportional to the mass of that body and to the rate of change of temperature.

3. Fourier's Law - Heat flows in the direction of decreasing temperature at a rate proportional to the gradient of the temperature, (so heat flow is positive when temperature gradient is negative.)

Notation - H = Heat: calorie, Joule, ... ; t = time: second, ...; T = Temperature: $^\circ C$, $^\circ F$, ... ; m = Mass: gram, slug, ... ; L = length: cm, ft, ... ; ρ = Density = $\frac{\text{mass}}{\text{volume}}$: $\frac{\text{gram}}{cm^3}$, ... ; c = Heat Capacity per unit of Mass: $\frac{\text{cal}}{\text{gram}\,^\circ C}$, ... ; κ ("kappa") = Thermal Conductivity: $\frac{\text{cal}}{sec\,cm\,^\circ C}$, ... ; k (small letter k) = Diffusivity = $\frac{\kappa}{c\,\rho}$: $\frac{cm^2}{sec}$.

Consider a slice of the rod which lies between x and $x+\Delta x$. Let $q(x,\,t)$ be the rate of heat flow at point x and time t: $\frac{\text{cal}}{sec\,cm^2}$, ..., and assume q is positive when heat flow to the right. Let $u(x,t)$ be the temperature at point x and time t: $^\circ C$, ...

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The rate at which heat is entering the slice through the surface at x with area A is $Aq(x,t)$: $\frac{\text{cal}}{sec}$, ..., and the rate at which heat is leaving the slice through the surface at $x+\Delta x$ is $Aq(x+\Delta x,\,t)$: $\frac{\text{cal}}{sec}$, ...

The heat stored in the slice is $c \underbrace{\rho\, A\, \Delta x}_{\text{mass}} \underbrace{\frac{\partial u}{\partial t}(x,\, t)}_{\substack{\text{rate of}\\ \text{change}\\ \text{of temp}}}:\ \frac{cal}{sec},\ \ldots$.

If the rate of the heat generated per unit of volume is g: $\frac{cal}{sec\, cm^3}$, ..., then rate at which heat is generated in the slice is $A\, \Delta x\, g$: $\frac{cal}{sec}$, ...

Thus, by the law of Conservation of Energy we have

$$A\, q(x,\, t) + A\, \Delta x\, g = A\, q(x + \Delta x,\, t) + c\, \rho\, A\, \Delta x\, \frac{\partial u}{\partial t}\,,\ \frac{cal}{sec},\ \text{and so}$$

$$\frac{q(x + \Delta x, t) - q(x,\, t)}{\Delta x} - g = -c\, \rho\, \frac{\partial u}{\partial t}$$

Taking the limit of both sides as $\Delta x \to 0$ we get

$$\frac{\partial q}{\partial x} - g = -c\, \rho\, \frac{\partial u}{\partial t}.$$

By the Fourier's law $q = -\kappa \frac{\partial u}{\partial x}$, and so $\frac{\partial q}{\partial x} = -\kappa \frac{\partial^2 u}{\partial x^2}$. Thus

$$\frac{\partial^2 u}{\partial x^2} + \frac{g}{\kappa} = \frac{c\, \rho}{\kappa} \frac{\partial u}{\partial t},\ t > 0.$$

If no heat is generated, $g = 0$, and letting $k = \frac{\kappa}{c\, \rho}$ we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k} \frac{\partial u}{\partial t},\ 0 < x < a,\ t > 0.$$

Now, $\frac{\partial^2 u}{\partial x^2} = \frac{1}{k} \frac{\partial u}{\partial t}$, $0 < x < a$, $t > 0$, describes the temperature u in a rod of length a with uniform properties and cross section, in which no heat is generated, and whose cylindrical surface is insulated. This equation has many solutions: $u(x,\, t) = x^2 + 2kt$, $u(x,\, t) = e^{-kt} \sin x$. We want to have a unique solution, therefore we place auxiliary conditions on our PDE:

1. The initial temperature distribution in the rod, (called initial condition, I.C., or initial value, I.V.).

2. What is happening at the ends of the rod, (boundary condition, B.C., or boundary value, B.V.)?

I.C.: $u(x,\, 0) = f(x)$, $0 < x < a$.
B.C.:

1. Dirichlet B.C. (also called B.C. of the 1st kind)
   $u(0,\, t) = T_0$, $u(a,\, t) = T_1$, $t > 0$ (fixed, end temperatures)

2. Neumann B.C. (also called B.C. of the 2nd kind)
   $\frac{\partial u}{\partial x}(0,\, t) = \phi_1(t)$, $\frac{\partial u}{\partial x}(a,\, t) = \phi_2(t)$, $t > 0$
   ϕ_1 or $\phi_2 = 0$ corresponds to an insulated surface at the end $x = 0$ or $x = a$, respectively.

3. Robin B.C. (also called B.C. of the 3rd kind)
   $c_1 u(0,\, t) + d_1 \frac{\partial u}{\partial x}(0,\, t) = \psi_1(t)$, $t > 0$
   $c_2 u(a,\, t) + d_2 \frac{\partial u}{\partial x}(a,\, t) = \psi_2(t)$, $t > 0$

For convection at the end $x = a$, $h\, u(a, t) + \kappa\, \frac{\partial u}{\partial x}(a,\, t) = h\, T_1(t)$, where h is convection coefficient: $\frac{cal}{cm^2\, sec\, ^\circ C}$, ..., and $T_1(t)$ is the temperature of the medium surrounding the end at $x = a$. Similarly for convection at the end $x = 0$, $h\,u(0, t) - \kappa\,\frac{\partial u}{\partial x}(0,\,t) = h\,T_0(t)$. *Classroom discussion!*

If a B.C. involves more than one boundary point, it is called a mixed boundary condition. For example: $u(0,\,t) = u(a,\,t)$ and $\frac{\partial u}{\partial x}(0,\,t) = \frac{\partial u}{\partial x}(a,\,t)$. Of course, there are many more possible boundary conditions.

An Initial Value - Boundary Value Problem:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}, && 0 < x < a,\, t > 0\\
&u(0,\,t) = T_0, && t > 0\\
&hu(a,\,t) + \kappa\frac{\partial u}{\partial x}(a,\,t) = h\,T_1, && t > 0\\
&u(x,\,0) = f(x), && 0 < x < a
\end{aligned}$$

There is exactly one and only one solution to a complete I.V. - B.V. problem.

2.2 Steady-State Temperatures

The steady-state temperature distribution is a time independent function $v(x)$ which is a solution of the time independent heat equation that satisfies the B.C.'s.

Physically, when heat conduction through a body is left undisturbed for a long time, the variation of the temperature with respect to time dies out and we achieve steady-state temperature distribution. In this case we expect

$$\lim_{t\to\infty} u(x,\,t) = v(x) \text{ and } \lim_{t\to\infty}\frac{\partial u}{\partial t} = 0.$$

Examples: State and solve (find) the steady-state problem corresponding to each of the following.

1.
$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}, && 0 < x < a,\, t > 0\\
&u(0,\,t) = T_0,\ u(a,\,t) = T_1, && t > 0\\
&u(x,\,0) = f(x), && 0 < x < a
\end{aligned}$$

The S-S problem is

$$\begin{aligned}
&\frac{d^2 v}{dx^2} = 0, && 0 < x < a\\
&v(0) = T_0,\ v(a) = T_1
\end{aligned}$$

and its solution is $v(x) = (T_1 - T_0)\frac{x}{a} + T_0$. *Classroom discussion!*

2. $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{k}\dfrac{\partial u}{\partial t}, \qquad 0 < x < a,\ t > 0$

$u(0,\,t) = u(a,\,t), \quad t > 0$

$u(x,\,0) = f(x), \qquad 0 < x < a$

The S-S problem is

$$\frac{d^2 v}{dx^2} = 0, \qquad 0 < x < a$$
$$v(0) = v(a)$$

and its solution is $v(x) = B$. *Classroom discussion!*

In the last example, the mathematical solution of the S-S problem is not unique. However, due to physical considerations there will be only one acceptable S-S solution!

The transient temperature distribution is the difference between the temperature $u(x,\,t)$ and the steady-state temperature $v(x)$: $w(x,\,t) = u(x,\,t) - v(x)$. Of course, it is called transient since physically we expect it to die out as $t \to \infty$.

Example. State the problem satisfied by the transient temperature distribution for the example 1 above.

The transient problem is

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{k}\frac{\partial w}{\partial t}, \qquad 0 < x < a,\ t > 0$$
$$w(0,\,t) = 0,\ w(a,\,t) = 0, \quad t > 0$$
$$w(x,\,0) = f(x) - v(x), \qquad 0 < x < a.$$

Classroom discussion!

Therefore, if we find the steady-state temperature and the transient temperature, then their sum will be the solution of the heat equation. Notice that the PDE and the B.C.'s which the transient temperature satisfies are linear and homogeneous.

2.3 Example: Fixed End Temperatures

We want to solve

$$\begin{aligned} &\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}, && 0 < x < a,\, t > 0 \\ &u(0,\, t) = T_0,\, u(a,\, t) = T_1, && t > 0 \text{ (Fixed, End Temperatures)} \\ &u(x,\, 0) = f(x), && 0 < x < a. \end{aligned}$$

The solution $u(x, t)$ is the sum of the steady-state solution $v(x)$ and the transient solution $w(x, t)$. The S-S problem is

$$\begin{aligned} &\frac{d^2 v}{dx^2} = 0, && 0 < x < a \\ &v(0) = T_0,\, v(a) = T_1 \end{aligned}$$

with the solution $v(x) = (T_1 - T_0)\frac{x}{a} + T_0$. The transient problem is

$$\begin{aligned} &\frac{\partial^2 w}{\partial x^2} = \frac{1}{k}\frac{\partial w}{\partial t}, && 0 < x < a,\, t > 0 \\ &w(0,\, t) = 0,\, w(a,\, t) = 0, && t > 0 \\ &w(x,\, 0) = f(x) - v(x), && 0 < x < a. \end{aligned}$$

we will solve the transient problem by the method of separation of variables (also called the product method and the Fourier's method). For this method to work it is essential to have a linear homogeneous PDE and boundary conditions of type 1, 2 or 3, to be homogeneous.

Assume $w(x, t) = \phi(x)h(t)$. Plug into PDE to get $\frac{\phi''(x)}{\phi(x)} = \frac{1}{k}\frac{h'(t)}{h(t)}$. Since the left-hand side is a function of x and the right-hand side is a function of t, then this equality can only hold if the two sides have the same constant value, say $-\lambda$. (The use of minus sign "$-$" is due to the fact that we will always have $-\lambda \leq 0$!) Plug w into the boundary conditions to get $\phi(0) = \phi(a) = 0$ since otherwise $h(t) = 0$, resulting in $w(x, t) = 0$ which is not acceptable unless $w(x, 0) = f(x) - v(x) = 0$. *Classroom discussion!*

We know have two problems to solve.

$$\frac{h'(t)}{h(t)} = -\lambda k, \quad t > 0 \qquad\qquad \begin{array}{l} \phi''(x) = -\lambda\phi(x), \quad 0 < x < a \\ \phi(0) = \phi(a) = 0 \end{array}$$

The first problem is a separable ODE and its solution is $h(t) = c\,e^{-\lambda kt}$. *Classroom discussion!*

The second problem is a second order linear ODE with constant coefficients. Its characteristic equation is $r^2 = -\lambda$ and its solution will depend on the sign of λ.

I. $\lambda < 0$. Then $r = \pm\sqrt{-\lambda}$ and we will get $\phi(x) = 0$ resulting in $w(x,\, t) = 0$. *Classroom discussion!*

II. $\lambda = 0$. Then $\phi(x) = 0$ resulting in $w(x,\, t) = 0$. *Classroom discussion!*

III. $\lambda > 0$. Let $\lambda = \mu^2$ with $\mu > 0$. Then $r = \pm\mu\, i$ and we will get $\phi(x) = \sin\mu x$ with $\mu = \frac{n\pi}{a}$ for $n = 1,\, 2,\, \cdots$. *Classroom discussion!*

Set $\mu_n = \frac{n\pi}{a}$, $\phi_n(x) = \sin\mu_n x$ and $h_n(t) = e^{-\mu_n^2 kt}$ for $n = 1,\, 2,\, \cdots$. Then each of the functions $w_n(x,\, t) = \sin\mu_n x\, e^{-\mu_n^2 kt}$ is a transient solution. Since in the transient problem, the PDE and boundary conditions are linear and homogeneous, any finite linear combination of w_n's will also be a solution of the transient problem. This is called the superposition principle. Using the superposition principle, we expect $w(x,\, t) = \sum_{n=1}^{\infty} b_n w_n(x,\, t) = \sum_{n=1}^{\infty} b_n \sin \mu_n x\, e^{-\mu_n^2 kt}$ to satisfy the PDE and boundary conditions of the transient problem. What is left to do is to find b_n's such that $w(x,\, 0) = f(x) - v(x)$ for $0 < x < a$.

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a} = f(x) - v(x),\ 0 < x < a$$

Classroom discussion!

If $f(x)$ is sectionally continuous, then F. sine series of $f(x) - v(x)$ is unique (because $\overline{f_o}(x) - \overline{v_o}(x)$ will be sectionally continuous since $v(x)$ is continuous). Therefore, b_n's are the F. sine series coefficients of the function $f(x) - v(x)$, $0 < x < a$:

$$b_n = \frac{2}{a} \int_0^a (f(x) - v(x)) \sin \frac{n\pi x}{a}\, dx.$$

Finally,

$$u(x,\, t) = T_0 + (T_1 - T_0)\frac{x}{a} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a}\, e^{-\frac{n^2\pi^2}{a^2}kt} \text{ with } b_n = \frac{2}{a}\int_0^a \left[f(x) - \left(T_0 + (T_1 - T_0)\frac{x}{a}\right)\right] \sin \frac{n\pi x}{a}\, dx.$$

Remarks: 1. For a graph of $u(x,\, t)$, see your book.
2. If f is not continuous, but sectionally smooth, then $u(x,\, 0) = \frac{1}{2}(f(x^-) + f(x^+))$ at all x values at which f has a hole or a jump.

Questions: 1. Does this infinite sum converge?
2. Does the u we have found satisfy the PDE, boundary conditions and initial conditions?
3. Is this solution unique?

The positive answer to these three questions is called "Mathematical Justification". In this class, we will call Mathematical Justification the proof for positive answer to the first two questions.

2.4 Example: Insulated Bar

We want to solve

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}, && 0 < x < a,\ t > 0\\
&\frac{\partial u}{\partial x}(0,\, t) = \frac{\partial u}{\partial x}(a,\, t) = 0, && t > 0 \text{ (Insulated Ends)}\\
&u(x,\, 0) = f(x), && 0 < x < a.
\end{aligned}$$

This is a (linear) homogeneous PDE with (linear) homogeneous B.C.'s, therefore S-S temperature is not needed. Assume $u(x,\, t) = \phi(x)h(t)$. Plug into PDE and boundary conditions to get the following two problems. *Classroom discussion!*

$$\frac{h'(t)}{h(t)} = -\lambda k, \quad t > 0 \qquad\qquad \begin{array}{l} \phi''(x) = -\lambda\phi(x), \quad 0 < x < a \\ \phi'(0) = \phi'(a) = 0 \end{array}$$

The first problem is a separable ODE and its solution is $h(t) = c\,e^{\lambda k t}$. *Classroom discussion!*

The second problem is a second order linear ODE with constant coefficients. Its characteristic equation is $r^2 = -\lambda$. Now, the solution will depend on the sign of λ.

I. $\lambda < 0$. Then $r = \pm\sqrt{-\lambda}$ and we will get $\phi(x) = 0$ resulting in $u(x,\, t) = 0$. *Classroom discussion!*

II. $\lambda = 0$. Then $\phi(x) =$ Constant resulting in $u(x,\, t) =$ Constant. *Classroom discussion!*

III. $\lambda > 0$. Let $\lambda = \mu^2$ with $\mu > 0$. Then $r = \pm\mu\, i$ and we will get $\phi(x) = \cos\mu x$ with $\mu = \frac{n\pi}{a}$ for $n = 1,\, 2,\, \cdots$. *Classroom discussion!*

Remark. Your book uses $-\lambda^2$ in place of $-\lambda$. That is, it assumes, a priori, that $-\lambda \leq 0$. We will not make this assumption. First, we will solve the problem, as above, then we will find the sign of λ using the Sturm-Liouville theorem, and eventually just use the result from the Review, Identities, Formulas and Theorems.

Set $\mu_n = \frac{n\pi}{a}$, $\phi_n(x) = \cos\mu_n x$ and $h_n(t) = e^{-\mu_n^2 kt}$ for $n = 0, 1, \cdots$. Then each of the functions $u_n(x, t) = \cos\mu_n x\, e^{-\mu_n^2 kt}$ is a solution. Notice this includes the case $\lambda = 0$, for which $u_0(x, t) = 1$ is, indeed, a solution. However, it is easier to treat the case $\lambda = 0$ or $u_0(x, t) = 1$, separately. Since the PDE and boundary conditions are linear and homogeneous, by the superposition principle, we expect $u(x, t) = a_0(1) + \sum_{n=1}^{\infty} a_n u_n(x, t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\mu_n x\, e^{-\mu_n^2 kt}$. What is left to do is to find a_0 and a_n's such that $u(x, 0) = f(x)$ for $0 < x < a$.

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{a} = f(x),\ 0 < x < a$$

Classroom discussion!

If $f(x)$ is sectionally continuous, then F. cosine series of $f(x)$ is unique (because $\overline{f_e}(x)$ will be sectionally continuous). Therefore, a_n's are the F. cosine series coefficients of the function $f(x)$, $0 < x < a$:

$$a_0 = \frac{1}{a}\int_0^a f(x)\,dx \text{ and } a_n = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}\,dx.$$

Finally,

$$u(x, t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{a}\, e^{-\frac{n^2\pi^2}{a^2}kt} \text{ with } a_0 \text{ and } a_n \text{ as above.}$$

Note. For a graph of $u(x, t)$, see your book.

Question. What happens to $u(x, 0) = f(x)$, if f is sectionally smooth but not continuous?

Exercises: 1. Show that $\lim_{t\to\infty} u(x, t) = a_0 = \frac{1}{a}\int_0^a f(x)\,dx$. Hint: You may interchange the order of the summation and that of the limit and assume $\lim_{n\to\infty} a_n = 0$.
2. Do the mathematical justification for this problem. Hint: See the last mathematical justification

problem and assume $\lim_{n\to\infty} a_n = 0$.
3. Let f be an even $2a$-periodic sectionally continuous function with F. cosine series coefficients a_0, a_n, $n = 1, 2 \cdots$. Show that $\lim\limits_{n\to\infty} a_n = 0$. Hint: Start with $f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{a}$ and $\int_0^a f^2(x)\,dx = \int_0^a a_0\, f(x)\,dx + \sum_{n=1}^{\infty} a_n \int_0^{\infty} f(x)\cos\frac{n\pi x}{a}\,dx$. This results in another form of the Parseval's equality.

2.5 Example: Different Boundary Conditions

First, we will discuss F. series of particular extensions of a function $f(x)$, defined on $(0, a)$, to the interval $(0, 2a)$ whose F. sine and cosine series will be of the form $\sum\limits_{n=1}^{\infty} b_n \sin\frac{(2n-1)\pi x}{2a}$ and $\sum\limits_{n=1}^{\infty} a_n \cos\frac{(2n-1)\pi x}{2a}$, respectively. Then, we will use them to solve certain I.V.-B.V. problems with mixed boundary conditions.

Examples: 1. Show that if f is a sectionally continuous function and $f(x) = \sum\limits_{n=1}^{\infty} b_n \sin\frac{(2n-1)\pi x}{2a}$ for $0 < x < a$, then $b_n = \frac{2}{a}\displaystyle\int_0^a f(x)\sin\frac{(2n-1)\pi x}{2a}\,dx$.

Let m be a positive integer. Multiply both sides by $\sin\frac{(2m-1)\pi x}{2a}$ and integrate with respect to x from $x = 0$ to $x = a$. Interchange the order of summation and integration and use the orthogonal properties of sine functions. *Classroom discussion!*

2. Let f be an arbitrary sectionally smooth and continuous function on $0 < x < a$. Show that $f(x) = \sum\limits_{n=1}^{\infty} b_n \sin\frac{(2n-1)\pi x}{2a}$ where $b_n = \frac{2}{a}\displaystyle\int_0^a f(x)\sin\frac{(2n-1)\pi x}{2a}\,dx$.

First extend f to the interval $(0, 2a)$ by reflecting it about the line $x = a$. Call this function F: $F(x) = \begin{cases} f(x), & 0 < x < a \\ f(2a - x), & a < x < 2a \end{cases}$. Then find the F. sine series coefficients of F and simplify to get an integral over the interval $0 < x < a$. Apply the convergence theorem of the F. sine series of F on the interval $0 < x < a$, in which $F(x) = f(x)$. *Classroom discussion!*

Exercises: 1. Show that if $f(x) = \sum_{n=1}^{\infty} a_n \cos \frac{(2n-1)\pi x}{2a}$ for $0 < x < a$, then $a_n = \frac{2}{a}\int_0^a f(x)\cos\frac{(2n-1)\pi x}{2a}\,dx$.

Hints: Multiply both sides by $\cos\frac{(2m-1)\pi x}{2a}$, where m is a positive integer, and integrate both sides with respect to x from $x = 0$ to $x = a$. Interchange the order of integration and summation. Use the orthogonal properties of cosine functions.

2. Let f be an arbitrary sectionally smooth and continuous function on $0 < x < a$. Show that $f(x) = \sum_{n=1}^{\infty} a_n \cos\frac{(2n-1)\pi x}{2a}$ where $a_n = \frac{2}{a}\int_0^a f(x)\cos\frac{(2n-1)\pi x}{2a}\,dx$.

Hints: First extend f to $F(x) = \begin{cases} f(x), & 0 < x < a \\ -f(2a-x), & a < x < 2a \end{cases}$ on $0 < x < 2a$. Then find the F. cosine series coefficients of F and simplify to get an integral over the interval $0 < x < a$. Apply the convergence theorem of the F. cosine series of F on the interval $0 < x < a$, in which $F(x) = f(x)$.

Remark. If $f(x)$, $0 < x < a$, is sectionally continuous, then its F.S. of the form $\sum_{n=1}^{\infty} b_n \sin\frac{(2n-1)\pi x}{2a}$ or $\sum_{n=1}^{\infty} a_n \cos\frac{(2n-1)\pi x}{2a}$ are unique. This is due to the uniqueness of F. series of periodic, sectionally continuous functions.

Now, let's solve the following I.V.-B.V. problem satisfied by the temperature in a uniform rod with initial temperature distribution $f(x)$, one end at fixed temperature T_0, and insulated at the other end.

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}, && 0 < x < a,\ t > 0 \\
&u(0,\,t) = T_0, && t > 0 \\
&\frac{\partial u}{\partial x}(a,\,t) = 0, && t > 0 \\
&u(x,\,0) = f(x), && 0 < x < a
\end{aligned}$$

We will solve this problem and also state the steps needed to solve a general heat equation with initial and boundary conditions.

1. If the PDE, a B.C., or both are not homogeneous (assuming both are already linear), find the steady-state temperature distribution $v(x)$. If this step is not needed go to step 3.

$$\frac{d^2v}{dx^2} = 0, \qquad 0 < x < a$$
$$v(0) = T_0$$
$$\frac{dv}{dx}(a) = 0$$

The S-S temperature distribution is $v(x) = T_0$. *Classroom discussion!*

2. Determine the I.V.-B.V. problem satisfied by the transient temperature distribution $w(x,\, t)$. *Classroom discussion!*

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{k}\frac{\partial w}{\partial t}, \qquad 0 < x < a,\ t > 0$$
$$w(0,\, t) = \frac{\partial w}{\partial x}(a,\, t) = 0, \quad t > 0$$
$$w(x,\, 0) = f(x) - T_0, \qquad 0 < x < a$$

3. Use the method of separation of variables.

   (a) Write $w(x,\, t) = \phi(x)h(t)$ (u in place of w, if step 1 was not needed) and plug it into the PDE and boundary conditions. The PDE will reduce to two ODE's, using a constant of separation, say $-\lambda$. One ODE involves T and is first order and the other is second order and involves ϕ. The boundary conditions will reduce to boundary conditions for the second order ODE. *Classroom discussion!*

$$\frac{h'(t)}{h(t)} = -\lambda k, \quad t > 0 \qquad\qquad \begin{aligned} &\phi''(x) = -\lambda\phi(x), \quad 0 < x < a \\ &\phi(0) = 0 \\ &\phi'(a) = 0 \end{aligned}$$

(b) Solve for T. *Classroom discussion!*

$$h(t) = C\, e^{-\lambda k t}$$

(c) Solve for ϕ, by considering negative, zero, and positive values of λ. In this step we will also find values of the constant of separation. This step will become shorter later on by use of the Sturm-Liouville theorem, and eventually will be done instantly by use of the result from Review, Identities, Formulas and Theorems.

i. $\lambda < 0$. *Classroom discussion!*

$\phi(x) \equiv 0 \Longrightarrow w(x,\, t) \equiv 0$, not acceptable!

ii. $\lambda = 0$. *Classroom discussion!*

$\phi(x) \equiv 0 \Longrightarrow w(x,\, t) \equiv 0$, not acceptable!

iii. $\lambda > 0$. Let $\lambda = \mu^2$ with $\mu > 0$. *Classroom discussion!*

$\mu = \frac{(2n-1)\pi}{2a}$, $\phi(x) = C_2 \sin \mu x$ for $n = 1, 2, \cdots$.

(d) Label the solutions: $\mu_n = \frac{(2n-1)\pi}{2a}$, $x_n(x) = \sin \mu_n x$, $h_n(t) = e^{-\mu_n^2 kt}$ and $w_n(x, t) = \phi_n(x) h_n(t)$, for $n = 1, 2, \cdots$. (We can use 1 for the constant coefficients. This step can be skipped or combined with the next one!)

(e) Use the superposition principle to make a linear combination of the solutions in the last step and find the constants using the initial condition.

$$w(x, t) = \sum_{n=1}^{\infty} b_n w_n(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{(2n-1)\pi x}{2a} e^{-\frac{(2n-1)^2 \pi^2}{4a^2} kt}$$

$$w(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{(2n-1)\pi x}{2a} = f(x) - T_0$$

Classroom discussion!

$$b_n = \frac{2}{a} \int_0^a (f(x) - T_0) \sin \frac{(2n-1)\pi x}{2a} \, dx$$

4. The solution is $u(x, t) = v(x) + w(x, t)$.

$$u(x, t) = T_0 + \sum_{n=1}^{\infty} b_n \sin \frac{(2n-1)\pi x}{2a} e^{-\frac{(2n-1)^2 \pi^2}{4a^2} kt} \text{ where } b_n = \frac{2}{a} \int_0^a (f(x) - T_0) \sin \frac{(2n-1)\pi x}{2a} \, dx$$

5. Mathematical Justification. (We will do this only if the problem specifically asks for it). *Classroom discussion!*

2.6 Example: Convection

We want to solve

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}, && 0 < x < a,\ t > 0 \\
&u(0,\,t) = T_0, && t > 0 \text{ (Fixed Temp } T_0 \text{ at the end } x = 0) \\
&\kappa\frac{\partial u}{\partial x}u(a,\,t) + hu(a,\,t) = hT_1, && t > 0 \text{ (Convection to a medium at Temp } T_1 \text{ at the end } x = a) \\
&u(x,\,0) = f(x), && 0 < x < a
\end{aligned}
$$

with k, κ and h positive constants.

The solution is $u(x,\,t) = v(x) + w(x,\,t)$ where

$$
\begin{aligned}
&\frac{d^2 v}{dx^2} = 0, && 0 < x < a \quad \text{and} && \frac{\partial^2 w}{\partial x^2} = \frac{1}{k}\frac{\partial w}{\partial t}, && 0 < x < a,\ t > 0 \\
&v(0) = T_0 && && w(0,\,t) = 0, && t > 0 \\
&\kappa v'(a) + hv(a) = hT_1 && && \kappa\frac{\partial w}{\partial x}(a,\,t) + hw(a,\,t) = 0, && t > 0 \\
& && && w(x,\,0) = f(x) - v(x), && 0 < x < a.
\end{aligned}
$$

Classroom discussion!

$v(x) = T_0 + \frac{h(T_1 - T_0)}{\kappa + ha} x.$

Let $w(x,\, t) = \phi(x)h(t)$ and apply the method of separation of variables. Discuss the three cases for the constant of separation. *Classroom discussion!*

For $n = 1,\, 2,\, \cdots,\, \lambda = -\mu_n^2$ where μ_n's are the positive solutions of $\tan \mu a = -\frac{\kappa}{h}\mu$ and $h_n(t) = e^{-\mu_n^2 kt}$, $\phi_n(x) = \sin \mu_n x$ and $w_n(x,\, t) = \phi_n(x)h_n(t)$. Use the superposition principle and apply the initial condition.

$$w(x,\,t) = \sum_{n=1}^{\infty} b_n w_n(x,\,t) = \sum_{n=1}^{\infty} b_n \, \sin \mu_n x \, e^{-\mu_n^2 kt}$$
$$w(x,\,0) = \sum_{n=1}^{\infty} b_n \, \sin \mu_n x = f(x) - v(x)$$

Show orthogonality of functions $\{\sin \mu_n x\}_{n=1}^{\infty}$ and use it to find the constant b_n's. *Classroom discussion!*

$$u(x,\,t) = T_0 + \tfrac{h(T_1 - T_0)}{\kappa + ha} x + \sum_{n=1}^{\infty} b_n \, \sin \mu_n x \, e^{-\mu_n^2 kt} \text{ where } b_n = \frac{\displaystyle\int_0^a \left(f(x) - T_0 - \tfrac{h(T_1 - T_0)}{\kappa + ha} x \right) \sin \mu_n x \, dx}{\displaystyle\int_0^a \sin^2 \mu_n x \, dx}$$

Exercise. Show that $\displaystyle\int_0^a \sin^2 \mu_n x \, dx = \frac{a}{2} + \frac{\kappa}{2h} \cos^2 \mu_n a.$

2.7 Sturm-Liouville Problem

Definition. The Sturm-Liouville (S-L) problem, or S-L eingenvalue problem (EVP) is

$$\begin{aligned}&\frac{d}{dx}\left[s(x)\,\tfrac{d\phi}{dx}\right]-q(x)\phi+\lambda p(x)\phi=0,\quad l<x<r\\&\alpha_1\phi(l)-\alpha_2\phi'(l)=0\\&\beta_1\phi(r)+\beta_2\phi'(r)=0\end{aligned}\quad\text{or}\quad\begin{aligned}&\frac{d}{dx}\left[s(x)\,\tfrac{d\phi}{dx}\right]-q(x)\phi+\lambda p(x)\phi=0,\quad l<x<r\\&\phi(l)=\phi(r)\\&s(l)\phi'(l)=s(r)\phi'(r)\end{aligned}$$

where

a. $s(x)$, $s'(x)$, $q(x)$ and $p(x)$ are continuous for $l \le x \le r$,

b. $s(x) > 0$ and $p(x) > 0$ for $l \le x \le r$,

c. $\alpha_1^2+\alpha_2^2 > 0$ (or, α_1 and α_2 are not both zero) and $\beta_1^2+\beta_2^2 > 0$ (or, β_1 and β_2 are not both zero), and

d. The parameter λ occurs only where shown.

Remark. Your textbook uses "λ^2" in place of "λ". We will never do that because it implies, in our notation, that $\lambda \ge 0$. This is something that must be proven!

Notes: 1. The above problem with the boundary conditions on the left side is called a Regular S-L EVP.
2. The above problem with the boundary conditions on the right side (periodic B.C.'s) is called a Irregular S-L EVP.
3. If $s(x)$, $s'(x)$, $q(x)$ and $p(x)$ are continuous only on $l < x < r$, or either $s(x)$ or $p(x)$ is positive only on $l < x < r$, then above problem with the boundary conditions on the left side is called a Singular S-L EVP.

Definition. The values of λ for which the S-L EVP has a nonzero solution are called eigenvalues and the corresponding nonzero solutions ϕ are called eigenfunctions.

The following two theorems are for the Regular S-L EVP: $\alpha_1\phi(l)-\alpha_2\phi'(l)=0$, $\beta_1\phi(r)+\beta_2\phi'(r)=0$.

Theorem 1. Consider the Regular Sturm-Liouville Problem.

a. The eigenvalues λ are real-valued.

b. There are infinitely many different eigenvalues $\lambda_1, \lambda_2, \cdots$ and an infinite number of eigenfunctions $\phi_1, \phi_2, \cdots$ corresponding to them.

c. The eigenfunctions are unique up to a constant multiple (If ϕ and ψ are two eigenfunctions corresponding to the same eigenvalue λ, then $\phi = c\psi$ for some nonzero constant c.)

d. If $n \ne m$, the eigenfunctions ϕ_n and ϕ_m are orthogonal with weight function $p(x)$: $\displaystyle\int_l^r \phi_n(x)\phi_m(x)p(x)\,dx = 0$ for $n \ne m$.

Proof of Orthogonality of Eigenfunctions - *Classroom discussion!*

Theorem 2. Consider the Regular Sturm-Liouville Problem.

a. $\lim_{n\to\infty} \lambda_n = \infty$.

b. If the eigenvalues are numbered in order $\lambda_1 < \lambda_2 < \cdots$, then the eigenfunction ϕ_n corresponding to λ_n has exactly $n-1$ zeros in the interval $l < x < r$ (endpoints excluded).

c. If $q(x) \geq 0$, $\alpha_1\alpha_2 \geq 0$ and $\beta_1\beta_2 \geq 0$, then all eigenvalues λ_n are nonnegative.

Proof of $\lambda \geq 0$ - *Classroom discussion!*

In the above proof we derived the following result.

Theorem 3. (Rayleigh Quotient) If $\dfrac{d}{dx}\left[s(x)\,\frac{d\phi}{dx}\right] - q(x)\phi + \lambda p(x)\phi = 0$, then

$$\lambda = \frac{-s(x)\phi(x)\,\frac{d\phi}{dx}\Big|_l^r + \displaystyle\int_l^r \left[s(x)(\tfrac{d\phi}{dx})^2 + q(x)\phi^2(x)\right]\,dx}{\displaystyle\int_l^r \phi^2(x)p(x)\,dx}.$$

Remark. If $\phi'(x) \not\equiv 0$, or $\phi(x)$ is not a constant function, then $\lambda > 0$.

To show this, we need the fact that for a continuous function f with $f(x) \not\equiv 0$ and $f(x) \geq 0$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)\,dx > 0$. (Alternatively, we could use the fact that for a continuous nonnegative function f with $\displaystyle\int_a^b f(x)\,dx = 0$, then $f(x) = 0$ for $a \leq x \leq b$.) *Classroom discussion!*

Theorem 4. For Irregular Sturm-Liouville problem (periodic boundary conditions $\phi(l) = \phi(r)$ and $s(l)\phi'(l) = s(r)\phi'(r)$), theorems 1 and 2 and the above remark hold, except for the uniqueness of the eigenfunctions.

Example. Show that the eigenvalues λ of the problem
$$\begin{aligned}&\phi''(x) = -\lambda\phi(x), \quad 0 < x < a\\ &\phi(0) = 0\\ &\kappa\phi'(a) + h\phi(a) = 0\end{aligned}$$
are positive valued, where $\kappa > 0$ and $h > 0$.

Classroom discussion!

Exercises: 1. Prove eigenfunctions of the problem
$$\begin{aligned}&\phi''(x) = -\lambda\phi(x), \quad 0 < x < a\\ &\phi(0) = \phi'(a) = 0\end{aligned}$$
are orthogonal: $\displaystyle\int_0^a \phi_n(x)\phi_m(x)\,dx = 0$ for $n \neq m$. (Do not just quote the S-L theorem!)

2. By use of the S-L theorem, find the exact value of the eigenvalues of the problem
$$\begin{aligned}&u''(x) = -\lambda u(x), \quad 0 < x < \pi\ .\\ &u(0) = u(\pi) = 0\end{aligned}$$

Theorem 5. (Generalized Fourier Series) Let $\phi_1, \cdots, \phi_n$ be eigenfunctions of the Regular Sturm-Liouville problem with $\alpha_1\alpha_2 \geq 0$ and $\beta_1\beta_2 \geq 0$. If $f(x)$ is a sectionally smooth function on the interval $l \leq x \leq r$, then $\displaystyle\sum_{n=1}^{\infty} c_n\,\phi_n(x) = \frac{1}{2}(f(x^-) + f(x^+))$ for $l < x < r$ where $c_n = \dfrac{\int_l^r f(x)\phi_n(x)p(x)\,dx}{\int_l^r \phi_n^2(x)\,p(x)\,dx}$.

Note. This theorem is in the section 2.8 of your textbook.

Exercise. Derive the formula for the constants in the last theorem. Suppose $f(x)$ is sectionally continuous and $f(x) = \displaystyle\sum_{n=1}^{\infty} c_n\,\phi_n(x)$, for $l < x < r$. Show that $c_n = \dfrac{\int_l^r f(x)\phi_n(x)p(x)\,dx}{\int_l^r \phi_n^2(x)\,p(x)\,dx}$. You may interchange the order of summation and integration.

2.9 Generalities on the Heat Conduction Problem

We want to solve the following I.V.-B.V. problem satisfied by the temperature in a nonuniform rod.

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\kappa(x)\frac{\partial u}{\partial x}\right) = \rho(x)c(x)\frac{\partial u}{\partial t}, && l < x < r,\, t > 0 \\
&\alpha_1 u(l,\,t) - \alpha_2\frac{\partial u}{\partial x}(l,\,t) = c_1, && t > 0,\, \alpha_1\alpha_2 \geq 0 \\
&\beta_1 u(r,\,t) + \beta_2\frac{\partial u}{\partial x}(r,\,t) = c_2, && t > 0,\, \beta_1\beta_2 \geq 0 \\
&u(x,\,0) = f(x), && l < x < r
\end{aligned}$$

The values (zero or positive) of the constants α's and β's correspond to fixed temperature, insulated or convection at the ends of the rod. *Classroom discussion!*

If not both $c_1 = 0$ and $c_2 = 0$, we must find the S-S temperature distribution $v(x)$, which is the solution of the following.

$$\begin{aligned}
&\frac{d}{dx}\left(\kappa(x)\frac{dv}{dx}\right) = 0, && l < x < r \\
&\alpha_1 v(l) - \alpha_2\frac{dv}{dx}(l) = c_1, && \alpha_1\alpha_2 \geq 0 \\
&\beta_1 v(r) + \beta_2\frac{du}{dx}(r) = c_2, && \beta_1\beta_2 \geq 0
\end{aligned}$$

Now, we discuss all cases except the case of both ends being insulated ($\alpha_1 = \beta_1 = 0$ and $c_1 = c_2 = 0$). We have seen that case before. In that case, $\lambda = 0$ was an eigenvalue of the S-L problem. So, we assume either α_1, β_1, or both are positive. Then

$$v(x) = \int_l^x \tfrac{A}{\kappa(\xi)}\,d\xi + B \text{ where } A \text{ and } B \text{ are solutions of the system } \begin{cases} \alpha_1 B - \alpha_2\frac{A}{\kappa(l)} = c_1 \\ \beta_1\left[\int_l^r \frac{A}{\kappa(\xi)}\,d\xi + B\right] + \beta_2\frac{A}{\kappa(r)} = c_2 \end{cases}.$$

Classroom discussion!

For ease of notation, let $\overline{\kappa} = \frac{1}{r-l}\int_l^r \kappa(x)\,dx$, $\overline{\rho} = \frac{1}{r-l}\int_l^r \rho(x)\,dx$, and $\overline{c} = \frac{1}{r-l}\int_l^r c(x)\,dx$; the average values of $\kappa(x)$, $\rho(x)$, and $c(x)$, over the interval $l \le x \le r$, respectively. Define $s(x) = \frac{\kappa(x)}{\overline{\kappa}}$ and $p(x) = \frac{\rho(x)c(x)}{\overline{\rho}\overline{c}}$, which are dimensionless, and let $k = \frac{\overline{\kappa}}{\overline{\rho}\overline{c}}$.

The transient temperature $w(x,,t)$ (or $u(x,\,t)$, if $c_1 = c_2 = 0$) is the solution of the following problem.

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(s(x)\frac{\partial w}{\partial x}\right) = \frac{1}{k}p(x)\frac{\partial w}{\partial t}, && l < x < r,\ t > 0\\
&\alpha_1 w(l,\,t) - \alpha_2\frac{\partial w}{\partial x}(l,\,t) = 0, && t > 0,\ \alpha_1\alpha_2 \ge 0\\
&\beta_1 w(r,\,t) + \beta_2\frac{\partial w}{\partial x}(r,\,t) = 0, && t > 0,\ \beta_1\beta_2 \ge 0\\
&w(x,\,0) = f(x) - v(x), && l < x < r
\end{aligned}$$

Classroom discussion!

Now, apply the method of separation of variables by assuming $w(x,\,t) = \phi(x)h(t)$. This will result in the following ODE and Regular S-L EVP.

$$\frac{h'(t)}{h(t)} = -\lambda k,\ t > 0 \qquad \begin{aligned}&\frac{d}{dx}\left[s(x)\,\frac{d\phi}{dx}\right] = -\lambda p(x)\phi, \quad l < x < r\\ &\alpha_1\phi(l) - \alpha_2\phi'(l) = 0\\ &\beta_1\phi(r) + \beta_2\phi'(r) = 0\end{aligned}$$

Classroom discussion!

Then $h(t) = e^{-\lambda k t}$. (We can use 1 for the constant of integration. Why?). It is easy to see that no nonzero constant function can be a solution for ϕ. (Show it!) *Classroom discussion!*

So according to our S-L theorems we have the following.

1. There are an infinite number of positive eigenvalues; $0 < \lambda_1 < \lambda_2 < \cdots$.
2. For each eigenvalue, there is a unique (up to a constant multiple) eigenfunction, ϕ_n, $n = 1, 2, \cdots$.
3. Eigenfunctions are orthogonal with weight function $p(x)$: $\displaystyle\int_l^r \phi_n(x)\phi_m(x)p(x)\,dx = 0$ for $n \neq m$.
4. If $f(x)$ is a sectionally smooth function on the interval $l \leq x \leq r$, then $\displaystyle\sum_{n=1}^{\infty} c_n\,\phi_n(x) = \frac{1}{2}(f(x^-)+f(x^+))$ for $l < x < r$ where $c_n = \dfrac{\int_l^r f(x)\phi_n(x)p(x)\,dx}{\int_l^r \phi_n^2(x)\,p(x)\,dx}$.

Set $w_n(x,\,t) = \phi_n(x)e^{-\lambda_n k t}$ and $w(x,\,t) = \displaystyle\sum_{n=1}^{\infty} c_n w_n(x,\,t) = \sum_{n=1}^{\infty} c_n\phi_n(x)e^{-\lambda_n k t}$. Find constant c_n so that $w(x,\,0) = \displaystyle\sum_{n=1}^{\infty} c_n\phi_n(x) = f(x) - v(x)$ for $l < x < r$.

Therefore, $u(x,\,t) = v(x) + \displaystyle\sum_{n=1}^{\infty} c_n\phi_n(x)e^{-\lambda_n k t}$ where $c_n = \dfrac{\int_l^r (f(x)-v(x))\phi_n(x)p(x)\,dx}{\int_l^r \phi_n^2(x)\,p(x)\,dx}$ and k, $p(x)$ and $v(x)$ as stated before.

Remarks: 1. If $f(x)$ is sectionally smooth, then $u(x, 0) = \frac{1}{2}(f(x^-) + f(x^+))$, while if $f(x)$ is also continuous, then $u(x, 0) = f(x)$.

2. $\lim_{t\to\infty} u(x, t) = v(x)$. For any fixed value t_1, the series $\sum_{n=1}^{\infty} c_n \phi_n(x) e^{-\lambda_n k t_1}$ converges uniformly, thus the solution $u(x, t_1)$ is a continuous function (in x) even though the initial condition $f(x)$ might not have been a continuous function.

Exercise. Show that for the above solution $\lim_{t\to\infty} u(x, t) = v(x)$. Hints: You may interchange the order of the limit and summation. Assume c_n's are bounded.

Chapter 3

The Wave Equation

3.1 The Vibrating String

We want to obtain the equation governing the motion of a flexible, taut string of finite length and with fixed endpoints after being put into motion by an initial force.

We will make the following assumptions.

1. The string is uniform: uniform cross section, uniform density, ...
2. The motion takes place entirely in one plane, and in that plane each particle moves at right angle to the equilibrium position of the string. *Classroom discussion!*

3. The string is perfectly flexible, that is; the tension at any point on the string is tangent to midline of the string at that point

We will make use of the following.

1. Newton's First Law of Motion - Some of forces on a particle in equilibrium is zero.
2. Newton's Second Law of Motion - $F = ma$.

Notation- L = length: cm, ft, ...; t = time: second, ...; m = Mass: gram, slug, ...; ρ = Linear Density = $\frac{\text{mass}}{\text{length}}$: $\frac{\text{gram}}{cm}$, $\frac{\text{kg}}{m}$, $\frac{\text{slug}}{ft}$

Consider a portion of the string which lies between x and $x + \Delta x$. The portions of the string to the right and left of our element exert forces on it which causes acceleration. Let $u(x, t)$ be the displacement of the string, at point x and time t, from the equilibrium; cm, ft, ...

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Let $T(x)$ and $T(x+\Delta x)$ be the tensions at the end x and $x+\Delta x$, respectively; $dyn = \frac{\text{gram}\, cm}{sec^2}$, $N = \frac{kg\, m}{sec^2}$, lb, ... Since our string only moves vertically, then the sum of forces in the horizontal direction must be zero (Newton's 1st Law of Motion). Hence

$$-T(x)\cos\alpha + T(x+\Delta x)\cos\beta = 0,\ \frac{\text{gram}\, cm}{sec^2},\ \Longrightarrow$$
$$T(x)\cos\alpha = T(x+\Delta x)\cos\beta$$

Since this will hold for every x and Δx, then we can assume each side of the above is the same constant.

$$T(x)\cos\alpha = T(x+\Delta x)\cos\beta = T \text{ constant} \Longrightarrow$$
$$T(x) = \frac{T}{\cos\alpha},\ T(x+\Delta x) = \frac{T}{\cos\beta}$$

Using Newton's 2nd Law of Motion in the vertical direction we get the following.

$$\underbrace{-T(x)\sin\alpha + T(x+\Delta x)\sin\beta - mg}_{F} = \underbrace{m\frac{\partial^2 u}{\partial t^2}(x,\,t)}_{m\,a}, \quad \frac{\text{gram}\,cm}{sec^2}$$

Divide both sides by $\cos\alpha$, use $m = \rho\,\Delta x$ and the above identities to get

$$-T\tan\alpha + T\tan\beta - \rho\,\Delta x\, g = \rho\,\Delta x\,\tfrac{\partial^2 u}{\partial t^2}(x,\,t).$$

Now, $\tan\alpha$ and $\tan\beta$ are the slopes of the strings at x and $x+\Delta x$, respectively; that is, $\tan\alpha = \frac{\partial u}{\partial x}(x,\,t)$ and $\tan\beta = \frac{\partial u}{\partial x}(x+\Delta x,\,t)$. Using these and dividing both sides by Δx we get the following.

$$T\,\frac{\frac{\partial u}{\partial x}(x+\Delta x,\,t) - \frac{\partial u}{\partial x}(x,\,t)}{\Delta x} = \rho\,\Delta x\left(\frac{\partial^2 u}{\partial t^2}(x,\,t) + g\right)$$

Taking the limit of both sides as $\Delta x \to 0$ we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \frac{1}{c^2}\,g, \text{ where } c^2 = \tfrac{T}{\rho}.$$

Assuming c^2 is very large in comparison to g, then we can neglect the term $\frac{1}{c^2}\,g$. This gives the equation of the vibrating string, wave equation, in one-dimension,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},\ 0 < x < a,\ t > 0.$$

Classroom discussion!

For the vibrating string we have described here, the boundary conditions are zero displacement at the ends; $u(0,\,t) = u(a,\,t) = 0$. But to describe the motion of the string we must also specify the initial position, $u(x,\,0)$, and the initial velocity, $\frac{\partial u}{\partial t}(x,\,0)$. Therefore, the BV-IV problem for the string under our assumptions is the following.

$$\begin{array}{ll}
\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}, & 0 < x < a,\ t > 0 \\
u(0,\,t) = u(a,\,t) = 0, & t > 0 \\
u(x,\,0) = f(x), & 0 < x < a \\
\dfrac{\partial u}{\partial t}(x,\,0) = g(x), & 0 < x < a
\end{array}$$

3.2 Solution of the Vibrating String Problem

We want to solve

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < a,\ t > 0$$
$$u(0,\,t) = u(a,\,t) = 0, \quad t > 0 \text{ (Fixed Ends)}$$
$$u(x,\,0) = f(x), \qquad 0 < x < a \text{ (Initial Position)}$$
$$\frac{\partial u}{\partial t}(x,\,0) = g(x), \qquad 0 < x < a \text{ (Initial Velocity).}$$

Since the PDE and the boundary conditions are linear and homogeneous, we can apply the method of separation of variables.

Assume $u(x,\,t) = \phi(x)h(t)$. Plug into PDE and boundary conditions. Using $-\lambda$ as the constant of separation we will get the following S-L EVP and ODE. Notice that the second order linear ODE with the constant coefficient can not be solved until we know the value of λ. *Classroom discussion!*

$$\phi''(x) = -\lambda\phi(x), \quad 0 < x < a \qquad\qquad h''(t) + \lambda c^2 h(t) = 0, \quad t > 0$$
$$X(0) = X(a) = 0$$

It is easy to see that no nonzero constant function can be a solution for X. (Show it!) Therefore, by the S-L theorem, $\lambda > 0$. Let $\lambda = \mu^2$ with $\mu > 0$. The solution of the above EVP is $\mu = \frac{n\pi}{a}$, $\phi(x) = \sin\mu a = \sin\frac{n\pi x}{a}$ for $n = 1,\,2,\,\cdots$. And the corresponding solutions of the ODE are $h(t) = a\cos\mu ct + b\sin\mu ct = a\cos\frac{n\pi ct}{a} + b\sin\frac{n\pi ct}{a}$. *Classroom discussion!*

Let $\mu_n = \frac{n\pi}{a}$, $\phi_n(x) = \sin\frac{n\pi x}{a}$, $h_n(t) = a_n \cos\frac{n\pi ct}{a} + b_n \sin\frac{n\pi ct}{a}$ and $u_n(x,\, t) = \phi_n(x)h_n(t) = (a_n \cos\frac{n\pi ct}{a} + b_n \sin\frac{n\pi ct}{a}) \sin\frac{n\pi x}{a}$.

By the superposition principle

$$u(x,\, t) = \sum_{n=1}^{\infty} c_n u_n(x,\, t) = \sum_{n=1}^{\infty} (a_n \cos\tfrac{n\pi ct}{a} + b_n \sin\tfrac{n\pi ct}{a}) \sin\tfrac{n\pi x}{a}$$

where we have written $a_n c_n$ and $b_n c_n$ as a_n and b_n again, respectively.

Now, use the initial conditions to find the constants a_n and b_n.

$$u(x,\, 0) = \sum_{n=1}^{\infty} a_n \sin\tfrac{n\pi x}{a} = f(x),\ 0 < x < a$$

By the uniqueness of the F. series, assuming $f(x)$ is sectionally continuous, $a_n = \dfrac{2}{a}\displaystyle\int_0^a f(x) \sin\tfrac{n\pi x}{a}\, dx$.

Now, assuming we can differentiate the series with respect to t term-by-term, we get

$$\frac{\partial u}{\partial t}(x,\, t) = \sum_{n=1}^{\infty} (-a_n \tfrac{n\pi c}{a} \sin\tfrac{n\pi ct}{a} + b_n \tfrac{n\pi c}{a} \cos\tfrac{n\pi ct}{a}) \sin\tfrac{n\pi x}{a} \text{ and}$$
$$\frac{\partial u}{\partial t}(x,\, 0) = \sum_{n=1}^{\infty} b_n \tfrac{n\pi c}{a} \sin\tfrac{n\pi x}{a} = g(x),\ 0 < x < a$$

Again, by the uniqueness of the F. series, assuming $g(x)$ is sectionally continuous, $b_n = \dfrac{2}{n\pi c}\displaystyle\int_0^a g(x) \sin\tfrac{n\pi x}{a}\, dx$.
Thus

$$u(x,\, t) = \sum_{n=1}^{\infty} (a_n \cos\tfrac{n\pi ct}{a} + b_n \sin\tfrac{n\pi ct}{a}) \sin\tfrac{n\pi x}{a} \text{ with}$$
$$a_n = \frac{2}{a}\int_0^a f(x) \sin\tfrac{n\pi x}{a}\, dx \text{ and } b_n = \frac{2}{n\pi c}\int_0^a g(x) \sin\tfrac{n\pi x}{a}\, dx.$$

Of course, if the problem asks for it, we will do the step of mathematical justification.

Exercise. Show that $u(x,\, t) = \displaystyle\sum_{n=1}^{\infty} a_n \cos\tfrac{n\pi ct}{a} \sin\tfrac{n\pi x}{a}$ satisfies

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < a,\ t > 0$$
$$u(0,\,t) = u(a,\,t) = 0, \quad t > 0$$
$$u(x,\,0) = f(x), \qquad 0 < x < a$$
$$\frac{\partial u}{\partial t}(x,\,0) = 0, \qquad 0 < x < a$$

where a_n's are F. sine series coefficients of continuous and sectionally smooth, odd $2a$-periodic extension of f.

Now, let's consider the following specific example:

$$f(x) = \begin{cases} \frac{2x}{a}, & 0 \le x < \frac{a}{2} \\ 2 - \frac{2x}{a}, & \frac{a}{2} \le x < a \end{cases} \qquad g(x) = 0,\ 0 < x < a.$$

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That is, the string is lifted up one unit in the middle and then released.

Since $g(x) = 0$, we have $b_n = 0$ for $n = 1,\,2,\,\cdots$. Earlier we found that the F. sine series of f is $\overline{f_o}(x) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^2}\sin\frac{n\pi x}{a}$, the equality is due to the convergence theorem. Therefore, $a_n = \frac{8}{\pi^2}\frac{\sin\frac{n\pi}{2}}{n^2}$ for $n = 1,\,2,\,\cdots$. So,

$$u(x,\,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^2}\sin\frac{n\pi x}{a}\cos\frac{n\pi ct}{a}.$$

Using the identity $\sin a\cos b = \frac{1}{2}(\sin(a+b) + \sin(a-b))$, we get

$$u(x,\,t) = \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{8}{\pi^2}\frac{\sin\frac{n\pi}{2}}{n^2}\sin\frac{n\pi(x-ct)}{a} + \frac{8}{\pi^2}\frac{\sin\frac{n\pi}{2}}{n^2}\sin\frac{n\pi(x+ct)}{a}\right] = \frac{1}{2}\left[\overline{f_o}(x-ct) + \overline{f_o}(x+ct)\right].$$

Now, we can easily graph $u(x,\,t)$ at different times.

PUT THE GRAPHS HERE

3.3 D'Alembert's Solution

There is another simple way to solve the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$. We start by making a change of variables. Let $w = x + ct$ and $z = x - ct$. Think of u as a function of w and z. Rewrite the wave equation in terms of independent variables w and z. Using the chain rule we can show that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial w} + \frac{\partial u}{\partial z}$$
$$\frac{\partial u}{\partial t} = c\frac{\partial u}{\partial w} + c\frac{\partial u}{\partial z}$$
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial w^2} + 2\frac{\partial^2 w}{\partial w \partial z} + \frac{\partial^2 u}{\partial z^2}$$
$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial w^2} - 2c^2\frac{\partial^2 w}{\partial w \partial z} + c^2\frac{\partial^2 u}{\partial z^2}$$

Classroom discussion!

Therefore,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \Longrightarrow \cdots \Longrightarrow \frac{\partial^2 u}{\partial w \partial z} = 0.$$

Now,

$$\frac{\partial^2 u}{\partial w \partial z} = 0 \Longrightarrow \cdots \Longrightarrow u(x,\, t) = \phi(x + ct) + \psi(x - ct)$$

where ϕ and ψ are arbitrary twice differentiable functions. *Classroom discussion!*

This solution, $u(x,\, t) = \phi(x+ct) + \psi(x-ct)$ where ϕ and ψ are arbitrary twice differentiable functions, of the wave equation is called the d'Alembert's solution.

Exercise. Suppose ϕ and ψ are twice differentiable functions. Show that $u(x,\, t) = \phi(x+ct) + \psi(x-ct)$ satisfies the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$.

Now, let's solve the problem we solved earlier by using F. series by the d'Alembert's method.

$$\begin{array}{ll} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}, & 0 < x < a,\, t > 0 \\ u(0,\, t) = u(a,\, t) = 0, & t > 0 \\ u(x,\, 0) = f(x), & 0 < x < a \\ \dfrac{\partial u}{\partial t}(x,\, 0) = g(x), & 0 < x < a \end{array}$$

We look for a solution of the form $u(x,\, t) = \phi(x+ct) + \psi(x-ct)$. This function already satisfies the PDE. Now, we must find functions ϕ and ψ so that u satisfies the initial and boundary conditions. First, we will apply the initial conditions. Notice that $\frac{\partial u}{\partial t}(x,\, t) = c\,\phi'(x+ct) - c\,\psi'(x-ct)$.

$$\begin{array}{llll} u(x,\, 0) = f(x) & \Longrightarrow & \phi(x) + \psi(x) = f(x), & 0 < x < a \\ \dfrac{\partial u}{\partial t}(x,\, 0) = g(x) & \Longrightarrow & c\,\phi'(x) - c\,\psi'(x) = g(x), & 0 < x < a \end{array}$$

The solution of this system is

$\phi(x) = \frac{1}{2}(f(x) + G(x) + k)$ and

$\psi(x) = \frac{1}{2}(f(x) - G(x) - k)$, for $0 < x < a$, where $G(x) = \frac{1}{c}\displaystyle\int_0^x g(\xi)\, d\xi$ and k is a constant.

Classroom discussion!

Notice that these formulas give ϕ and ψ only on the interval $(0,\, a)$, and so we can not use them in $u(x,\, t) = \phi(x+ct) + \psi(x-ct)$ since the arguments $x \pm ct$ will not be in $(0,\, a)$ for large time t values. To overcome this problem, we will extend f and g to the entire real number line; call these new functions $\widetilde{f}$ and $\widetilde{g}$, respectively. So, $\phi(x) = \frac{1}{2}(\widetilde{f}(x) + \widetilde{G}(x) + k)$ and $\phi(x) = \frac{1}{2}(\widetilde{f}(x) - \widetilde{G}(x) - k)$. Now, apply the boundary conditions to figure out what type of extensions is appropriate.

$$\begin{aligned} u(0,\, t) = 0 \quad &\Longrightarrow \quad \left[\widetilde{f}(ct) + \widetilde{f}(-ct)\right] + \left[\widetilde{G}(ct) - \widetilde{G}(-ct)\right] = 0 \\ u(a,\, t) = 0 \quad &\Longrightarrow \quad \left[\widetilde{f}(a+ct) + \widetilde{f}(a-ct)\right] + \left[\widetilde{G}(a+ct) - \widetilde{G}(a-ct)\right] = 0 \end{aligned}$$

Classroom discussion!

Since functions f and g (or G) are independent of one another, these conditions hold only if

$$\begin{matrix} \widetilde{f}(ct) + \widetilde{f}(-ct) = 0 \\ \widetilde{f}(a+ct) + \widetilde{f}(a-ct) = 0 \end{matrix} \quad \text{and} \quad \begin{matrix} \widetilde{G}(ct) - \widetilde{G}(-ct) = 0 \\ \widetilde{G}(a+ct) - \widetilde{G}(a-ct) = 0 \end{matrix}.$$

The condition $\widetilde{f}(ct)+\widetilde{f}(-ct) = 0$ holds if $\widetilde{f}$ is an odd function, while the condition $\widetilde{G}(ct)-\widetilde{G}(-ct) = 0$ holds if $\widetilde{G}$ is an even function. The remaining two conditions $\widetilde{f}(a+ct)+\widetilde{f}(a-ct) = 0$ and $\widetilde{G}(a+ct)-\widetilde{G}(a-ct) = 0$ will also hold if both $\widetilde{f}$ and $\widetilde{G}$ are 2a-periodic functions. *Classroom discussion!*

Therefore, $\widetilde{f}(x) = \overline{f_o}(x)$ and $\widetilde{G}(x) = \overline{G_e}(x)$. Hence, $\phi(x) = \frac{1}{2}(\overline{f_o}(x) + \overline{G_e}(x) + k)$, and $\psi(x) = \frac{1}{2}(\overline{f_o}(x) - \overline{G_e}(x) - k)$, for $-\infty < x < \infty$. Finally,

$$u(x,\, t) = \tfrac{1}{2}\left[\overline{f_o}(x+ct) + \overline{f_o}(x-ct)\right] + \tfrac{1}{2}\left[\overline{G_e}(x+ct) + \overline{G_e}(x-ct)\right] \text{ where } G(x) = \tfrac{1}{c}\int_0^x g(\xi)\, d\xi.$$

Question. What extension of g results in $G(x) = \frac{1}{c}\int_0^x g(\xi)\, d\xi$ being an even $2a$-periodic function?

The answer to the above question, is in the following exercises.

Exercises: 1. Show that if $g(x)$ is an **even** (or odd) function, then $G(x) = \int_0^x g(\xi)\, d\xi$ is an **odd** (or even) function.
2. Show that if $g(x)$ is an odd $2a$-periodic function, then $\int_x^{x+2a} g(\xi)\, d\xi = 0$. Hint: Use the earlier result that $\int_c^{c+p} f(x)\, dx = \int_0^p f(x)\, dx$, for any p-periodic function f, twice; once for $c = x$ and then for $c = -a$.
3. Show that if $g(x)$ is an **odd** (or even) $2a$-periodic function, then $G(x) = \int_0^x g(\xi)\, d\xi$ is an **even** (or odd) $2a$-periodic function.

Now, we can write the above d'Alembert's solution in the following more informative way.

Exercise. Show that the above solution can be written as

$$u(x,\,t) = \frac{1}{2}\left[\,\overline{f_o}(x+ct) + \overline{f_o}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \overline{g_o}(\xi)\,d\xi.$$

Remarks: 1. Notice that the solution at the point $(x,\,t)$ depends on the initial conditions in the interval $[x-ct,\,x+ct]$. This interval is called the domain of dependence. PUT THE GRAPH HERE.

2. Said another way, the initial condition at $(x,\,0)$ influences the solution in the region between the lines $y = x - ct$ and $y = x + ct$. This region is called the region of influence. PUT THE GRAPH HERE.

To complete the relationships between a function and its extension, do the following exercise.

Exercise. Show that if differentiable function $f(x)$ is an **odd** (or even), $2a$-periodic function, then $f'(x)$ is an **even** (or odd), $2a$-periodic function. Hint: Start with $f(-x) = f(x)$ or $f(-x) = -f(x)$, and $f(x+2a) = f(x)$. Differentiate both sides.

The above result will be useful in the following exercise.

Exercise. Solve

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, && 0 < x < a,\ t > 0\\
&\frac{\partial u}{\partial x}(0,\,t) = \frac{\partial u}{\partial x}(a,\,t) = 0, && t > 0\\
&u(x,\,0) = f(x), && 0 < x < a\\
&\frac{\partial u}{\partial t}(x,\,0) = 0, && 0 < x < a
\end{aligned}$$

by the d'Almebert's method.

3.4 Generalities on the One-Dimensional Wave Equation

We want to solve the following I.V.-B.V. problem satisfied by the motion of a nonuniform string.

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(s(x)\frac{\partial u}{\partial x}\right) = \frac{p(x)}{c^2}\frac{\partial^2 u}{\partial t^2}, && l < x < r,\ t > 0\\
&\alpha_1 u(l,\,t) - \alpha_2\frac{\partial u}{\partial x}(l,\,t) = c_1, && t > 0,\ \alpha_1\alpha_2 \ge 0\\
&\beta_1 u(r,\,t) + \beta_2\frac{\partial u}{\partial x}(r,\,t) = c_2, && t > 0,\ \beta_1\beta_2 \ge 0\\
&u(x,\,0) = f(x), && l < x < r\\
&\frac{\partial u}{\partial t}(x,\,0) = g(x), && l < x < r
\end{aligned}$$

where $s(x)$, $s'(x)$ and $p(x)$ are continuous and both $s(x)$ and $p(x)$ are positive on $l \le x \le r$.

Suppose the string ends are attached to a spring-mass system and allowed to move only vertically in a slot. Let k_l and k_r be the spring constants at the ends $x = l$ and $x = r$, respectively. Let T_l and T_r be the tensions in the string at the end $x = l$ and $x = r$, respectively. PUT THE GRAPH HERE.

Then, the boundary condition at the end $x = l$ is $-k_l u(l, t) + T_l \frac{\partial u}{\partial x}(l, t) = m\frac{\partial^2 u}{\partial t^2}(l, t)$ while the boundary condition at the end $x = r$ is $k_r u(r, t) + T_r \frac{\partial u}{\partial x}(r, t) = m\frac{\partial^2 u}{\partial t^2}(l, t)$. If the mass is small, then $k_l u(l, t) - T_l \frac{\partial u}{\partial x}(l, t) = 0$ and $k_r u(r, t) - T_r \frac{\partial u}{\partial x}(r, t) = 0$. And if no spring is used (the ends move freely up or down in the vertical slots, without friction), then $\frac{\partial u}{\partial x}(l, t) = 0$ and $\frac{\partial u}{\partial x}(l, t) = 0$. This is called the free end boundary condition. The boundary conditions listed in our problem are the most general conditions. *Classroom discussion!*

Now, we discuss all cases except the case $\alpha_1 = \beta_1 = 0$. If not both $c_1 = 0$ and $c_2 = 0$, we look for a solution of the form $u(x, t) = v(x) + w(x, t)$. However, neither of the names"steady state solution" nor "transient solution" is appropriate. Here, $v(x)$ represents the equilibrium solution and is the solution of the following.

$$\begin{aligned}
&\frac{d}{dx}\left(s(x)\frac{dv}{dx}\right) = 0, \qquad l < x < r\\
&\alpha_1 v(l) - \alpha_2 \frac{dv}{dx}(l) = c_1, \quad \alpha_1\alpha_2 \ge 0\\
&\beta_1 v(r) + \beta_2 \frac{dv}{dx}(r) = c_2, \quad \beta_1\beta_2 \ge 0
\end{aligned}$$

We can solve this as we did for the heat equation.

$$v(x) = \int_l^x \frac{A}{s(\xi)}\,d\xi + B \text{ where } A \text{ and } B \text{ are solutions of the system } \begin{cases} \alpha_1 B - \alpha_2 \frac{A}{s(l)} = c_1 \\ \beta_1 \left[\int_l^r \frac{A}{s(\xi)}\,d\xi + B\right] + \beta_2 \frac{A}{s(r)} = c_2 \end{cases}.$$

The function $w(x, t)$ (or $u(x, t)$, if $c_1 = c_2 = 0$) is the solution of the following problem.

$$\frac{\partial}{\partial x}\left(s(x)\frac{\partial w}{\partial x}\right) = \frac{p(x)}{c^2}\frac{\partial w}{\partial t}, \quad l < x < r,\, t > 0$$
$$\alpha_1 w(l,\, t) - \alpha_2 \frac{\partial w}{\partial x}(l,\, t) = 0, \quad t > 0,\, \alpha_1\alpha_2 \geq 0$$
$$\beta_1 w(r,\, t) + \beta_2 \frac{\partial w}{\partial x}(r,\, t) = 0, \quad t > 0,\, \beta_1\beta_2 \geq 0$$
$$w(x,\, 0) = f(x) - v(x), \quad l < x < r$$
$$\frac{\partial w}{\partial t}(x,\, 0) = g(x), \quad l < x < r$$

Classroom discussion!

Now, apply the method of separation of variables by assuming $w(x,\, t) = \phi(x)h(t)$. This will result in the following ODE and Regular S-L EVP.

$$\frac{d}{dx}\left[s(x)\,\tfrac{dX}{dx}\right] = -\lambda p(x)X, \quad l < x < r \qquad\qquad h''(t) + \lambda c^2 h(t) = 0,\, t > 0$$
$$\alpha_1 X(l) - \alpha_2 X'(l) = 0$$
$$\beta_1 X(r) + \beta_2 X'(r) = 0$$

Classroom discussion!

Since we have assumed either α_1, β_1, or both are positive, it is easy to see that no nonzero constant function can be a solution for X. (Show it!) In the case $\alpha_1 = \beta_1 = 0$, $\lambda = 0$ will be an eigenvalue of the S-L problem. *Classroom discussion!*

So, according to S-L theorems we have the following.

1. There are an infinite number of positive eigenvalues: $0 < \lambda_1 < \lambda_2 < \cdots$.
2. For each eigenvalue, there is a unique (up to a constant multiple) eigenfunction, ϕ_n, $n = 1,\, 2,\, \cdots$.
3. Eigenfunctions are orthogonal with weight function $p(x)$: $\displaystyle\int_l^r \phi_n(x)\phi_m(x)p(x)\, dx = 0$ for $n \neq m$.
4. If $f(x)$ is a sectionally smooth function on the interval $l \leq x \leq r$, then $\displaystyle\sum_{n=1}^{\infty} c_n\, \phi_n(x) = \tfrac{1}{2}(f(x^-) + f(x^+))$

for $l < x < r$ where $c_n = \dfrac{\int_l^r f(x)\phi_n(x)p(x)\,dx}{\int_l^r \phi_n^2(x)\,p(x)\,dx}$.

Since $\lambda > 0$, the solution of the ODE is $h(t) = a\cos\sqrt{\lambda}ct + b\sin\sqrt{\lambda}ct$. For $n = 1, 2, \cdots$, set $\lambda_n = \mu_n^2$ with $\mu_n > 0$, $\phi_n(x) = \phi_n(x)$, $h_n(t) = a_n\cos\mu_n ct + b_n\sin\mu_n ct$, and $w_n(x, t) = (a_n\cos\mu_n ct + b_n\sin\mu_n ct)\phi_n(x)$, then $w(x, t) = \sum_{n=1}^{\infty}(a_n\cos\mu_n ct + b_n\sin\mu_n ct)\phi_n(x)$. Assuming we can differentiate the series with respect to t term-by-term, we have $\frac{\partial w}{\partial t}(x, t) = \sum_{n=1}^{\infty}(-a_n\mu_n c\sin\mu_n ct + b_n\mu_n c\cos\mu_n ct)\phi_n(x)$. Find constant a_n and b_n so that initial conditions are satisfied. *Classroom discussion!*

$$w(x, 0) = \sum_{n=1}^{\infty} a_n\phi_n(x) = f(x) - v(x), \quad l < x < r$$
$$\frac{\partial w}{\partial t}(x, 0) = \sum_{n=1}^{\infty} b_n\mu_n c\phi_n(x) = g(x), \qquad l < x < r$$

So,

$$a_n = \frac{\displaystyle\int_l^r (f(x) - v(x))\phi_n(x)p(x)\,dx}{\displaystyle\int_l^r \phi_n^2(x)\,p(x)\,dx} \text{ and } b_n = \frac{1}{\mu_n c}\frac{\displaystyle\int_l^r g(x)\phi_n(x)p(x)\,dx}{\displaystyle\int_l^r \phi_n^2(x)\,p(x)\,dx}.$$

Hence,

$$u(x,t) = \int_l^x \frac{A}{s(\xi)}\,d\xi + B + \sum_{n=1}^{\infty}(a_n\cos\mu_n ct + b_n\sin\mu_n ct)\phi_n(x)$$

where A, B, a_n and b_n are as stated before.

Remarks: 1. The mathematical justification step still remains and we will not do them here!
2. $\lim_{t\to\infty} u(x, t)$ does not exist.
3. There is no simple relationship between μ_n's. (In the case of the uniform string with fixed ends, we had $\mu_n = n\mu_1$.)
4. $u(x, t)$ is not periodic in time.

Chapter 4

The Potential Equation

4.1 Potential (Laplace) Equation

Definitions: 1. $\Delta u = \nabla^2 u = 0$ is called the potential or Laplace's equation.

Δu can be read as delta u or Laplacian of u. $\nabla^2 u$ is read as nabla-squared u or del-squared u. Nabla is the name for an ancient harp in middle east and is used more recently for ∇, due to its shape. Traditionally, the symbol ∇ was called del.

$$\begin{aligned} &\text{In one dimension,} && \Delta u = \nabla^2 u = \frac{d^2 u}{dx^2} \\ &\text{In two dimensions,} && \Delta u = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\ &\text{In } n \text{ dimensions,} && \Delta u = \nabla^2 u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} \end{aligned}$$

2. Solutions of the potential equation are called harmonic functions.

We can think of Laplace's equation as the time-independent (equilibrium) part of the heat or wave equation. Many physical phenomena are described by this equation.

Suppose $\nabla^2 u = 0$ in some region Ω (a line segment, a rectangle, etc.). We usually have one of the following three types of boundary conditions.

1. The value of u on the boundary of Ω, $\partial\Omega$, is specified; $u|_{\partial\Omega} = f(x)$.

2. The value of directional derivative of u along the outward pointing unit normal $\hat{n}$ on the boundary is given; $\frac{\partial u}{\partial n}|_{\partial\Omega} = f(x)$. Recall that $\frac{\partial u}{\partial n} = \nabla u \cdot \hat{n}$.

3. $\left(\alpha u + \beta \frac{\partial u}{\partial n}\right)|_{\partial\Omega} = f(x)$.

The boundary value problem consisting of the potential equation and

I. the B.C. 1 is called the Dirichlet's problem.

II. the B.C. 2 is called the Neumann's problem.

III. the B.C. 3 is called the Robin's problem.

Theorem 1. (Maximum Principle) Suppose $\nabla^2 u = 0$ on some open (does not contain its boundary), connected (one piece), bounded (contained in a box of finite dimensions) set Ω, or equivalently, $\nabla^2 u = 0$ on some bounded domain Ω. Then, if u is not constant, it must contain its maximum or minimum value on the boundary of Ω.

Example. Suppose $u(x, y)$ is a function defined on the open rectangle R and its boundary with $u|_{\partial R} = 1$, while at some point in R, the funcion value of u is 2. Then, by the Maximum Principle, we can not have $\nabla^2 u = 0$ in R. That is, u is not a harmonic function. PUT THE GRAPH HERE.

Theorem 2. Suppose Ω is as in the Maximum Principle theorem and

$$\begin{array}{ll} \nabla^2 u = 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{array}.$$

Then $u = 0$ in Ω.

Exercise. Prove the last theorem using the Maximum Principle.

Theorem 3. Suppose Ω is as in the Maximum Principle theorem. Then the solution of the Dirichlet's problem

$$\begin{array}{ll} \nabla^2 u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{array}$$

is unique.

Exercise. Using Theorem 2, prove the Theorem 3.

Remark. Solution of the Neumann's problem is not unique, since if u is a solution, then $u + c$ is also a solution, where c is any constant. Show it!

Now, consider the potential equation in two dimensions. We can write $\nabla^2 u$ in polar coordinates, as follows.

$x = r\cos\theta$

$y = r\sin\theta$

$r = \sqrt{x^2 + y^2}$

$\theta = \begin{cases} \tan^{-1}\frac{y}{x}, & x > 0 \\ \pi + \tan^{-1}\frac{y}{x}, & x < 0 \end{cases}$

$-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ and $\theta \neq \frac{\pi}{2}$, $r > 0$

PUT GRAPH HERE

If $x = 0$ and $y \neq 0$, then $r = |y|$ and $\theta = (\text{sign of } y)\frac{\pi}{2}$. If $x = y = 0$, then $r = 0$ and θ is arbitrary.

Using the chain rule we will find the partial derivatives of r and θ with respect to each of the variables x and y and use them to find the first and second partial derivatives of u with respect to the variables x and y.

$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2 + y^2)^{-\frac{1}{2}}\,2x = \cdots = \cos\theta$. Similarly, $\frac{\partial r}{\partial y} = \sin\theta$. $\frac{\partial \theta}{\partial x} = \frac{-\frac{y}{x^2}}{1+\frac{y^2}{x^2}} = \cdots = -\frac{\sin\theta}{r}$. Similarly, $\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$.

Classroom discussion!

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial x}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial u}{\partial x}) = \frac{\partial}{\partial r}(\frac{\partial u}{\partial x})\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}(\frac{\partial u}{\partial x})\frac{\partial \theta}{\partial x} = \cdots = \cos^2\theta\frac{\partial^2 u}{\partial r^2} - 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta}$

Similarly, $\frac{\partial^2 u}{\partial y^2} = \sin^2\theta\frac{\partial^2 u}{\partial r^2} + 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} - 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta}$. *Classroom discussion!*

After substituting in $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ and simplifying, we will get

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Classroom discussion!

4.2 Potential in a Rectangle

We want to solve the Dirichelt's problem in a rectangle. PUT THE GRAPH HERE.

$$\begin{aligned} &\nabla^2 u = 0, && 0 < x < a,\ 0 < y < b \\ &u(x, 0) = f_1(x),\ u(x, b) = f_2(x), && 0 < x < a \\ &u(0, y) = u(a, y) = 0, && 0 < y < b \end{aligned}$$

Since the PDE and two of the boundary conditions (on parallel sides) are linear and homogeneous, we can apply the method of separation of variables.

Assume $u(x, y) = X(x)Y(y)$. Plug into PDE and boundary conditions. Using $-\lambda$ as the constant of separation we will get the following S-L EVP and ODE. Notice that the second order linear ODE with the constant coefficient can not be solved until we know the value of λ. *Classroom discussion!*

$$X''(x) = -\lambda X(x), \quad 0 < x < a \qquad\qquad Y''(y) - \lambda Y(y) = 0, \quad 0 < y < b$$
$$X(0) = X(a) = 0$$

It is easy to see that no nonzero constant function can be a solution for X. (Show it!) Therefore, by the S-L theorem, $\lambda > 0$. Let $\lambda = \mu^2$ with $\mu > 0$. The solution of the above EVP is $\mu = \frac{n\pi}{a}$, $X(x) = \sin \mu a = \sin \frac{n\pi x}{a}$ for $n = 1, 2, \cdots$. And the corresponding solutions of the ODE are $Y(y) = A\,e^{\mu y} + B\,e^{-\mu y} = A\,e^{\frac{n\pi y}{a}} + B\,e^{-\frac{n\pi y}{a}}$. *Classroom discussion!*

At this stage it would be helpful to use the identity

$$A\,e^z + B\,e^{-z} = (A+B)\tfrac{e^z+e^{-z}}{2} + (A-B)\tfrac{e^z-e^{-z}}{2} = (A+B)\cosh z + (A-B)\sinh z$$

and using A and B, again, in place of $A+B$ and $A-B$, we can write $Y(y) = A \cosh \frac{n\pi y}{a} + B \sinh \frac{n\pi y}{a}$.

Remark. Another way of obtaining the solution of Y is to notice that $y_1 = \cosh \mu y$ and $y_2 = \sinh \mu y$ are two solutions of the ODE (Show it!) and they are also independent ($W(y_1, y_2) = y_1 y_2' - y_2 y_1' = \cdots \neq 0$), hence $Y(y) = A, \cosh \mu y + B \sinh \mu y$.

For $n = 1, 2, \cdots$, let $\mu_n = \frac{n\pi}{a}$, $X_n(x) = \sin \frac{n\pi x}{a}$, $Y_n(t) = A_n \cosh \frac{n\pi y}{a} + B_n \sinh \frac{n\pi y}{a}$ and $u_n(x, y) = X_n(x) Y_n(y) = (A_n \cosh \frac{n\pi y}{a} + B_n \sinh \frac{n\pi y}{a}) \sin \frac{n\pi x}{a}$.

By the superposition principle

$$u(x, y) = \sum_{n=1}^{\infty} d_n u_n(x, t) = \sum_{n=1}^{\infty} (a_n \cosh \tfrac{n\pi y}{a} + c_n \sinh \tfrac{n\pi y}{a}) \sin \tfrac{n\pi x}{a}$$

where we have written $d_n A_n$ and $d_n B_n$ as a_n and c_n, respectively.

Now, use the remaining two boundary conditions to find the constants a_n and c_n.

$$u(x, 0) = \sum_{n=1}^{\infty} a_n \sin \tfrac{n\pi x}{a} = f_1(x),\ 0 < x < a$$

By the uniqueness of the F. series, assuming f_1 is sectionally continuous, $a_n = \dfrac{2}{a} \displaystyle\int_0^a f_1(x) \sin \tfrac{n\pi x}{a}\, dx$.

$$u(x, b) = \sum_{n=1}^{\infty} (a_n \cosh \tfrac{n\pi b}{a} + c_n \sinh \tfrac{n\pi b}{a}) \sin \tfrac{n\pi x}{a} = f_2(x),\ 0 < x < a$$

By the uniqueness of the F. series, assuming f_2 is sectionally continuous, $c_n = \frac{1}{\sinh \frac{n\pi b}{a}} b_n - \frac{\cosh \frac{n\pi b}{a}}{\sinh \frac{n\pi b}{a}} a_n$ where $b_n = \dfrac{2}{a} \displaystyle\int_0^a f_2(x) \sin \tfrac{n\pi x}{a}\, dx$.

Thus,

$$u(x, y) = \sum_{n=1}^{\infty} \left[\tfrac{b_n}{\sinh \frac{n\pi b}{a}} \sinh \tfrac{n\pi y}{a} + a_n \left(\cosh \tfrac{n\pi y}{a} - \tfrac{\cosh \frac{n\pi b}{a}}{\sinh \frac{n\pi b}{a}} \sinh \tfrac{n\pi y}{a} \right) \right] \sin \tfrac{n\pi x}{a}.$$

Using a hyperbolic function identity, $\cosh \frac{n\pi y}{a} - \frac{\cosh \frac{n\pi b}{a}}{\sinh \frac{n\pi b}{a}} \sinh \frac{n\pi y}{a} = \frac{\sinh \frac{n\pi}{a}(b-y)}{\sinh \frac{n\pi b}{a}}$.
See Review, Identities, Formulas and Theorems.

Finally,

$$u(x, y) = \sum_{n=1}^{\infty} \left[\tfrac{a_n}{\sinh \frac{n\pi b}{a}} \sinh \tfrac{n\pi}{a}(b-y) + \tfrac{b_n}{\sinh \frac{n\pi b}{a}} \sinh \tfrac{n\pi y}{a} \right] \sin \tfrac{n\pi x}{a}, \text{ where}$$

$$a_n = \frac{2}{a} \int_0^a f_1(x) \sin \tfrac{n\pi x}{a}\, dx \text{ and } b_n = \frac{2}{a} \int_0^a f_2(x) \sin \tfrac{n\pi x}{a}\, dx.$$

Of course, if the problem asks for it, we will do the step of mathematical justification.

Remark. For solving $Y'' = \mu^2 Y$, we could have noticed that $y_1 = \sinh \mu y$ and $y_2 = \sinh \mu(b-y)$ are two independent solutions of it (Show it!) and, immediately, written $Y(y) = a \sinh \mu y + c \sinh \mu(b-y)$. This is the most convenient form of the solution Y since when we plug in $y = 0$ and $y = b$, one term of it is zero, making the calculation of the constants easier.

Now consider the following more general problem. PUT THE GRAPH HERE.

$$\begin{aligned} &\nabla^2 u = 0, && 0 < x < a,\, 0 < y < b \\ &u(x,0) = f_1(x),\, u(x,b) = f_2(x), && 0 < x < a \\ &u(0,y) = g_1(y),\, u(a,y) = g_2(y), && 0 < y < b \end{aligned}$$

Since we do not have homogeneous boundary conditions on two parallel sides, we break this problem into two such problems.

$$\begin{aligned} &\nabla^2 u_1 = 0, && 0 < x < a,\, 0 < y < b \\ &u_1(x,0) = f_1(x), && 0 < x < a \\ &u_1(x,b) = f_2(x), && 0 < x < a \\ &u_1(0,y) = 0, && 0 < y < b \\ &u_1(a,y) = 0, && 0 < y < b \end{aligned} \qquad \begin{aligned} &\nabla^2 u_2 = 0, && 0 < x < a,\, 0 < y < b \\ &u_2(x,0) = 0, && 0 < x < a \\ &u_2(x,b) = 0, && 0 < x < a \\ &u_2(0,y) = g_1(y), && 0 < y < b \\ &u_2(a,y) = g_2(y), && 0 < y < b \end{aligned}$$

Since the PDE and all boundary conditions are linear, it is easy to see that if u_1 and u_2 are solutions of these problems, then $u = u_1 + u_2$ is a solution of the original problem. (Show it!). We have already solved for u_1 and the solution for u_2 will be similar.

$$u_2(x,y) = \sum_{n=1}^{\infty} \left[\frac{A_n}{\sinh \frac{n\pi a}{b}} \sinh \frac{n\pi}{b}(a-x) + \frac{B_n}{\sinh \frac{n\pi a}{b}} \sinh \frac{n\pi x}{b} \right] \sin \frac{n\pi y}{b}, \text{ where}$$

$$A_n = \frac{2}{b}\int_0^b g_1(y) \sin \frac{n\pi y}{b}\, dy \text{ and } B_n = \frac{2}{b}\int_0^b g_2(x) \sin \frac{n\pi y}{b}\, dy.$$

Remark. Think about types of problems (PDE, boundary or initial conditions, and domain) we can apply the method of the separation of variables.

4.3 Further Examples for a Rectangle

We want to solve the following potential equation in a rectangle with mixed boundary conditions.

$$\begin{aligned} &\nabla^2 u = 0, && 0 < x < a,\, 0 < y < b \\ &u(x,0) = f_1(x),\, u(x,b) = f_2(x), && 0 < x < a \\ &\frac{\partial u}{\partial x}(0,y) = \frac{\partial u}{\partial x}(a,y) = 0, && 0 < y < b \end{aligned}$$

Since the PDE and two of the boundary conditions (on parallel sides) are linear and homogeneous, we can apply the method of separation of variables.

Assume $u(x,y) = X(x)Y(y)$. Plug into PDE and boundary conditions. Using $-\lambda$ as the constant of separation we will get the following S-L EVP and ODE. Notice that the second order linear ODE with the constant coefficient can not be solved until we know the value of λ. *Classroom discussion!*

$$X''(x) = -\lambda X(x), \quad 0 < x < a \qquad\qquad Y''(y) - \lambda Y(y) = 0, \quad 0 < y < b$$
$$X'(0) = X'(a) = 0$$

By the S-L theorem, $\lambda \geq 0$. If $\lambda = 0$, then the solution of $X''(x) = 0$, $X'(0) = X'(a) = 0$, is $X(x) = 1$ and the solution of $Y''(y) = 0$ is $y(y) = A + By$. Set $X_0(x) = 1$ and $Y_0(y) = A_0 + B_0\, y$, then $u_0(x,\, y) = X_0(y)Y_0(y) = A_0 + B_0\, y$. *Classroom discussion!*

For all other cases, $\lambda > 0$. Let $\lambda = \mu^2$ with $\mu > 0$. The solution of the above EVP is $\mu = \frac{n\pi}{a}$, $X(x) = \cos \mu x$ where $n = 1, 2, \cdots$. And the corresponding solutions of the ODE are $Y(y) = A \sinh \mu y + B \sinh \mu(b - y)$. *Classroom discussion!*

For $n = 1, 2, \cdots$, let $\mu_n = \frac{n\pi}{a}$, $X_n(x) = \cos \mu_n x$ and $Y_n(y) = A_n \sinh \mu_n y + B_n \sinh \mu_n(b - y)$, then

$u_n(x,\, y) = X_n(y)Y_n(y) = (A_n \sinh \frac{n\pi y}{a} + B_n \sinh \frac{n\pi}{a}(b-y)) \cos \frac{n\pi x}{a}$.

Remarks: 1. Since we will form a linear combination of all solutions $u(x,\, y) = X(x)Y(y)$, that is, multiply each by a constant and add them, we used one "1" for the constant in the solution of $X(x)$.
2. We chose to write $Y_n(y) = A_n \sinh \mu_n y + B_n \sinh \mu_n(b-y)$ in order to make future calculations for constants a bit easier; at $y=0$ and $y=b$, one of the terms is zero.

By the superposition principle,

$$u(x,\, y) = c_0 u_0(x,\, y) + \sum_{n=1}^{\infty} c_n u_n(x,\, y) = c_0 + d_0\, y + \sum_{n=1}^{\infty} (c_n \sinh \tfrac{n\pi y}{a} + d_n \sinh \tfrac{n\pi}{a}(b-y)) \cos \tfrac{n\pi x}{a}$$

where we have used c_n and d_n in place of $c_n A_n$ and $c_n B_n$, respectively, for $n = 0,\, 1,\, \cdots$.

Now, use the remaining two boundary conditions to find the constants c_n and d_n.

$$u(x,\, 0) = c_0 + \sum_{n=1}^{\infty} d_n \sinh \tfrac{n\pi b}{a} \cos \tfrac{n\pi x}{a} = f_1(x),\ 0 < x < a$$

By the uniqueness of the F. series, assuming f_1 is sectionally continuous,

$$c_0 = \frac{1}{a}\int_0^a f_1(x)\, dx \text{ and } d_n = \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a f_1(x) \cos \tfrac{n\pi x}{a}\, dx.$$

$$u(x,\, b) = c_0 + bd_0 + \sum_{n=1}^{\infty} c_n \sinh \tfrac{n\pi b}{a} \cos \tfrac{n\pi x}{a} = f_2(x),\ 0 < x < a$$

By the uniqueness of the F. series, assuming f_1 is sectionally continuous,

$$d_0 = \frac{1}{ab}\int_0^a (f_2(x) - c_0)\, dx \text{ and } c_n = \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a (f_2(x) - c_0) \cos \tfrac{n\pi x}{a}\, dx.$$

Classroom discussion!

Now, solve the following potential equation in a rectangle with another type of mixed boundary conditions.

$$\begin{aligned}
&\nabla^2 u = 0, && 0 < x < a,\ 0 < y < b\\
&\frac{\partial u}{\partial y}(x,\,0) = f(x), && 0 < x < a\\
&u(x,\,b) = 0, && 0 < x < a\\
&\frac{\partial u}{\partial x}(0,\,y) = 0, && 0 < y < b\\
&u(a,\,y) = g(y), && 0 < y < b
\end{aligned}$$

Since we do not have homogeneous boundary conditions on two parallel sides, we break this problem into two such problems.

$$\begin{aligned}
&\nabla^2 u_1 = 0, && 0 < x < a,\ 0 < y < b \qquad && \nabla^2 u_2 = 0, && 0 < x < a,\ 0 < y < b\\
&\frac{\partial u_1}{\partial y}(x,\,0) = 0, && 0 < x < a && \frac{\partial u_2}{\partial y}(x,\,0) = f(x), && 0 < x < a\\
&u_1(x,\,b) = 0, && 0 < x < a && u_2(x,\,b) = 0, && 0 < x < a\\
&\frac{\partial u_1}{\partial x}(0,\,y) = 0, && 0 < y < b && \frac{\partial u_2}{\partial x}(0,\,y) = 0, && 0 < y < b\\
&u_1(a,\,y) = g(y), && 0 < y < b && u_2(a,\,y) = 0, && 0 < y < b
\end{aligned}$$

Since the PDE and all boundary conditions are linear, it is easy to see that if u_1 and u_2 are solutions of these problems, then $u = u_1 + u_2$ is a solution of the original problem. (Show it!).

First, solve for u_1. Assume $u_1(x,\,y) = X(x)Y(y)$. Plug into PDE and boundary conditions to get the following S-L EVP and ODE with one boundary condition. *Classroom discussion!*

$$\begin{aligned}
&Y''(y) = -\lambda Y(y), \quad 0 < y < b \qquad && X''(x) - \lambda X(x) = 0, \quad 0 < x < a\\
&Y'(0) = Y(b) = 0 && X'(0) = 0
\end{aligned}$$

It is easy to see that no nonzero constant function can be a solution for Y. (Show it!) Therefore, by the S-L theorem, $\lambda > 0$. Let $\lambda = \mu^2$ with $\mu > 0$. The solution of the above EVP is $\mu = \frac{(2n-1)\pi}{2b}$, $Y(y) = c\cos\mu y$ where $n = 1,\,2,\,\cdots$. And the corresponding solutions of the ODE with one B. C. are $X(x) = d\cosh\mu x$. *Classroom discussion!*

For $n = 1,\, 2,\, \cdots$, let $\mu_n = \frac{(2n-1)\pi}{2b}$, $X_n(x) = \cosh \mu_n x$ and $Y_n(y) = \cos \mu_n y$, then $u_n(x,\, y) = X_n(y)Y_n(y) = \cosh \frac{(2n-1)\pi x}{2b} \cos \frac{(2n-1)\pi y}{2b}$.

Remarks: 1. Since we will form a linear combination of all solutions $u(x,\, y) = X(x)Y(y)$, that is, multiply each by a constant and add them, we used one "1" for the constants in the solutions.
2. We chose to write $X(x) = c \sinh \mu x + d \cosh \mu x$ in order to make future calculations a bit easier.

By the superposition principle,

$$u_1(x,\, y) = \sum_{n=1}^{\infty} a_n u_n(x,\, y) = \sum_{n=1}^{\infty} a_n \cosh \tfrac{(2n-1)\pi x}{2b} \cos \tfrac{(2n-1)\pi y}{2b}$$

Now, use the remaining boundary condition to find the constants a_n.

$$u_1(a,\, y) = \sum_{n=1}^{\infty} a_n \cosh \tfrac{(2n-1)\pi a}{2b} \cos \tfrac{(2n-1)\pi y}{2b} = g(y),\ 0 < y < b$$

By the uniqueness of the F. series, assuming g is sectionally continuous,

$$a_n = \frac{2}{b \cosh \frac{(2n-1)\pi a}{2b}} \int_0^b g(y) \cos \tfrac{(2n-1)\pi y}{2b}\, dy.$$

We will solve for u_2 in a similar fashion. Assume $u_2(x,\, y) = X(x)Y(y)$. Plug into PDE and boundary conditions to get the following S-L EVP and ODE with one boundary condition. *Classroom discussion!*

$$X''(x) = -\lambda X(x), \quad 0 < x < a \qquad\qquad Y''(y) - \lambda Y(y) = 0, \quad 0 < y < b$$
$$X'(0) = X(a) = 0 \qquad\qquad Y'(0) = 0$$

It is easy to see that no nonzero constant function can be a solution for X, therefore, by the S-L theorem, $\lambda > 0$. Let $\lambda = \nu^2$ with $\nu > 0$. Then $\nu = \frac{(2n-1)\pi}{2a}$, $X(x) = c\cos\nu x$ and $Y(y) = d\sinh\nu(b-y)$ for $n = 1, 2, \cdots$. *Classroom discussion!*

For $n = 1, 2, \cdots$, let $\nu_n = \frac{(2n-1)\pi}{2a}$, $X_n(x) = \cos\nu_n x$ and $Y_n(y) = \sinh\mu_n(b-y)$, then $u_n(x, y) = X_n(y)Y_n(y) = \cos\frac{(2n-1)\pi x}{2a}\sinh\frac{(2n-1)\pi}{2a}(b-y)$.

By the superposition principle,

$$u_2(x, y) = \sum_{n=1}^{\infty} b_n u_n(x, y) = \sum_{n=1}^{\infty} b_n \cos\frac{(2n-1)\pi x}{2a}\sinh\frac{(2n-1)\pi}{2a}(b-y)$$

Now, use the remaining boundary condition to find the constants b_n.

$$\frac{\partial u_2}{\partial y}(x,\, y) = \sum_{n=1}^{\infty} -\tfrac{(2n-1)\pi}{2a} b_n \cos\tfrac{(2n-1)\pi x}{2a} \cosh\tfrac{(2n-1)\pi}{2a}(b-y)$$

$$\frac{\partial u_2}{\partial y}(x,\, 0) = \sum_{n=1}^{\infty} -\tfrac{(2n-1)\pi}{2a} \cosh\tfrac{(2n-1)\pi b}{2a} b_n \cos\tfrac{(2n-1)\pi x}{2a} = f(x),\ 0<x<a$$

By the uniqueness of the F. series, assuming f is sectionally continuous,

$$b_n = -\frac{4}{(2n-1)\pi\cosh\frac{(2n-1)\pi b}{2a}} \int_0^a f(x)\cos\tfrac{(2n-1)\pi x}{2a}\, dx.$$

Finally,

$$u(x,\, y) = \sum_{n=1}^{\infty} a_n \cosh\tfrac{(2n-1)\pi x}{2b} \cos\tfrac{(2n-1)\pi y}{2b} + \sum_{n=1}^{\infty} b_n \cos\tfrac{(2n-1)\pi x}{2a} \sinh\tfrac{(2n-1)\pi}{2a}(b-y) \text{ where}$$

$$a_n = \frac{2}{b\cosh\frac{(2n-1)\pi a}{2b}} \int_0^b g(y)\cos\tfrac{(2n-1)\pi y}{2b}\, dy \text{ and } b_n = -\frac{4}{(2n-1)\pi\cosh\frac{(2n-1)\pi b}{2a}} \int_0^a f(x)\cos\tfrac{(2n-1)\pi x}{2a}\, dx.$$

4.5 Potential in a Disk

We want to solve the Dirichelt's problem in a disk. PUT THE GRAPH HERE.

$$\begin{aligned}
&\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, && 0<r<c,\ -\pi<\theta<\pi\\
&u(c,\, \theta) = f(\theta), && -\pi<\theta<\pi\\
&u(r,\, -\pi) = u(r,\, \pi), && 0<r<c\\
&\frac{\partial u}{\partial \theta}(r,\, -\pi) = \frac{\partial u}{\partial \theta}(r,\, \pi), && 0<r<c\\
&u(0,\, \theta) \text{ bounded}, && -\pi<\theta<\pi
\end{aligned}$$

We have used polar coordinates since the domain is a circle. In polar coordinates points $(r,\, \theta)$ and $(r,\, \theta{+}2\pi)$ are the same point, so we just need to use a 2π length for θ values. This also implies that the solution should be 2π periodic, since we must have $u(r,\, \theta) = u(r,\, \theta + 2\pi)$. We have chosen to use the interval $(-\pi,\, \pi)$ for θ since it matches with the way we defined F. series. Now, since $(r,\, -\pi) = (r,\, \pi)$, we should have the same solution values at those points. The conditions $u(r,\, -\pi) = u(r,\, \pi)$ and $\frac{\partial u}{\partial \theta}(r,\, -\pi) = \frac{\partial u}{\partial \theta}(r,\, \pi)$ ensures that 2π-periodic functions u and $\frac{\partial u}{\partial \theta}$ are continuous at $\theta = \pm\pi$. Of course, we are looking for a solution defined on the entire disk. Specifically, we will use that fact at the origin. This condition has been listed as $u(0,\, \theta)$ bounded. We could have also stated that $(0,\, \theta)$ should be in the domain of u, or $\lim_{r\to 0} u(r, \theta)$ must be a finite number.

Remark. Another option, in place of restricting θ, could have been to allow both u and f be 2π-periodic functions and $-\infty < \theta < \infty$. In that case, we still need the continuity and boundedness conditions.

Apply the method of separation of variables. Assume $u(r,\, \theta) = h(r)\phi(\theta)$. Plug into PDE and continuity and boundedness conditions to get the following irregular S-L EVP and ODE with an added boundedness condition. Note: I used $\phi(\theta)$ rather than $\Theta(\theta)$ for ease of hand writing. *Classroom discussion!*

$\phi''(\theta) = -\lambda\phi(\theta), \quad -\pi < \theta < \pi$	$r^2h''(r) + rh(r) - \lambda h(r) = 0, \quad 0 < r < c$
$\phi(-\pi) = \phi(\pi)$	$R(0)$ bounded
$\phi'(-\pi) = \phi'(\pi)$	

By the (Irregular) S-L theorem, $\lambda \geq 0$. If $\lambda = 0$, then $\phi(\theta) = c_1\theta + c_2$ and $\phi(-\pi) = \phi(\pi)$ implies that $c_1 = 0$. Thus $\phi(\theta) = c_2$ and the constant function also satisfies the 2nd boundary condition.

Now, assume $\lambda > 0$. Since we have not solved this EVP before, we will solve it here.

Characteristic Equation: $r^2 = -\lambda \Rightarrow r = \pm\sqrt{\lambda}\, i \quad \Longrightarrow \quad \phi(\theta) = c_1 \cos\sqrt{\lambda}\,\theta + c_2 \sin\sqrt{\lambda}\,\theta$

Continuity Condition for q: $\phi(-\pi) = \phi(\pi) \quad \Rightarrow \cdots \Rightarrow \quad 2c_2 \sin\sqrt{\lambda}\,\pi = 0$

Continuity Condition for ϕ': $\phi'(-\pi) = \phi'(\pi) \quad \Rightarrow \cdots \Rightarrow \quad 2c_1\sqrt{\lambda} \sin\sqrt{\lambda}\,\pi = 0$

Classroom discussion!

Since $\sqrt{\lambda} > 0$ and not both c_1 and c_2 can be zero (why?), we must have $\sin\sqrt{\lambda}\,\pi = 0$. Thus $\lambda = n^2$ for $n = 1, 2, \cdots$ and $\phi(\theta) = c_1 \cos n\theta + c_2 \sin n\theta$. *Classroom discussion!*

Remark. Notice that we can combine these two cases and write solutions as $\lambda = n^2$ and $\phi(\theta) = c_1 \cos n\theta + c_2 \sin n\theta$ for $n = 0,\, 1,\, \cdots$. (For n=0, $\lambda = 0$ and $\phi(\theta) = c_1$.)

For $\lambda = 0$, the solution of $r^2 h''(r) + rh(r) + \lambda h(r) = r\frac{d}{dr}\left(r\frac{dh}{dr}\right) = 0$ with $R(0)$ bounded is $h(r) = d_1$. *Classroom discussion!*

For $\lambda = n^2$, $n = 1,\, 2,\, \cdots$, the equation $r^2 h''(r) + rh(r) - n^2 h(r) = 0$ is the well-known Cauchy-Euler equation, whose solutions are of the form $h(r) = r^\alpha$. Plugging$h(r) = r^\alpha$, $h'(r) = \alpha r^{\alpha-1}$ and $h''(r) = \alpha(\alpha-1)r^{\alpha-2}$ into the ODE we will get $\alpha = \pm n$. The two linearly independent solutions are $h_1 = r^{-n}$ and $h_2 = r^n$ and the general solution is $h(r) = d_1 r^{-n} + d_2 r^n$. The condition $R(0)$ bounded implies that $h(r) = d_2 r^n$. See Review, Identities, Formulas and Theorems. *Classroom discussion!*

Remark. We can also combine these two cases and write solutions as $\phi(\theta)\, r^n$ for $n = 0,\, 1,\, \cdots$.

By the superposition principle,

$$u(r,\, \theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)\, r^n$$

Now, use the remaining boundary condition to find the constants.

$$u(c,\, \theta) = a_0 + \sum_{n=1}^{\infty} (c^n\, a_n \cos n\theta + c^n\, b_n \sin n\theta) = f(\theta),\ -\pi < \theta < \pi$$

By the uniqueness of the F. series and the convergence theorem, assuming f is continuous and sectionally smooth,

$$a_0 = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta,\ a_n = \tfrac{1}{\pi c^n}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta \text{ and } b_n = \tfrac{1}{\pi c^n}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta.$$

Notice the following property of this solution that can be generalized. At $r = 0$, $u(0,\, \theta) = a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(c,\, \theta)\, d\theta$. That is, the solution at the origin is the average value of the boundary condition on the circle of radius c, centered at the origin.

Theorem. (Mean Value Property) Suppose $\nabla^2 u = 0$ on some connected, open region Ω with smooth boundary. Then for any disk of radius c centered at the origin, which lies entirely in Ω, the value of u at the origin is the average value of u on the boundary of the disk, circle of radius c centered at the origin.

$$u(0,\, \theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(c,\, \theta)\, d\theta$$

Remark. This theorem is also true if the center of the disk is at any point P. The value of u at the point P is the average value of u on the boundary of the disk of radius c, centered at P. PUT THE GRAPH HERE.

Exercises: 1. Solve the above problem for $f(\theta) = 1$.
2. Solve the Laplace equation on the half-plane.

$$\begin{aligned}
&\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, && 0 < r < c,\, 0 < \theta < \pi \\
&u(c,\, \theta) = f(\theta), && 0 < \theta < \pi \\
&u(r,\, 0) = u(r, \pi) = 0, && 0 < r < c \\
&u(0,\, \theta) \text{ bounded}, && 0 < \theta < \pi
\end{aligned}$$

3. Solve the Laplace equation on the quarter-plane.

$$\begin{aligned}
&\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, && 0 < r < c,\, 0 < \theta < \tfrac{\pi}{2} \\
&u(c,\, \theta) = f(\theta), && 0 < \theta < \tfrac{\pi}{2} \\
&u(r,\, 0) = u(r, \tfrac{\pi}{2}) = 0, && 0 < r < c \\
&u(0,\, \theta) \text{ bounded}, && 0 < \theta < \tfrac{\pi}{2}
\end{aligned}$$

4.6 Classification of Partial Differential Equations (Classification and Limitations in the Course Textbook)

So far we have studies heat, wave and potential equations. Here is a summary of their qualitative features.

Equation	Features
Heat	Exponential behavior in time. Existence of a limiting (steady-state) solution. Smooth graph for $t > 0$.
Wave	Oscillatory (not always periodic) behavior in time. Retention of discontinuity for $t > 0$.
Potential	Smooth surface, Maximum Principle, Mean Value Property

The most general 2nd-order linear PDE is two variables is

$$A\,\frac{\partial^2 u}{\partial \xi^2} + B\,\frac{\partial^2 u}{\partial \xi \partial \eta} + C\,\frac{\partial^2 u}{\partial \eta^2} + D\,\frac{\partial u}{\partial \xi} + D\,\frac{\partial u}{\partial \eta} + F\,u + G = 0$$

where the coefficient $A,\, B,\, \cdots$ are, in general, functions of ξ and η. We classify these PDE's as follows:

Elliptic at $(\xi,\, \eta)$	if $B^2 - 4AC < 0$ at $(\xi,\, \eta)$
Parabolic at $(\xi,\, \eta)$	if $B^2 - 4AC = 0$ at $(\xi,\, \eta)$
Hyperbolic at $(\xi,\, \eta)$	if $B^2 - 4AC > 0$ at $(\xi,\, \eta)$

The potential equation is elliptic, the heat equation is parabolic and the wave equation is hyperbolic. (Check it!)

Example. Classify the Tricomi equation $y\,u_{xx} + u_{yy} = 0$.
Classroom discussion!

Canonical Forms of Second Order Linear PDE's -

Equation	Canonical Form
Elliptic	$u_{\xi\xi} + u_{\eta\eta} +$ terms with lower-order derivatives $= 0$
Parabolic	$u_{\xi\xi} +$ terms with lower-order derivatives $= 0$
Hyperbolic	$u_{\xi\eta} +$ terms with lower-order derivatives $= 0$ or $u_{\xi\xi} - u_{\eta\eta} +$ terms with lower-order derivatives $= 0$

Any of these three types of equations can be put in their canonical form, by simply making an appropriate change of coordinate system.

In this section our book also has a discussion on where we might be able to use the method of separation of variables. Read it!

Chapter 5

Problems in Several Dimensions

5.1 Two-Dimensional Heat and Wave Equations (Sections 5.1 and 5.2 in the course textbook)

Here I will merely state these equations and leave the derivation for you to read from the course textbook. The two dimensional heat equation is

$$\kappa \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \rho\, c\, \frac{\partial u}{\partial t} - g\,, \quad g \text{ is the heat generation rate}$$

$$\text{I.C.} \quad u(x,\, y,\, 0) = f(x,\, y)$$

and the boundary conditions can be any of the three types we have seen before, or mixed ones.

If no heat is generated, $g \equiv 0$, and letting $k = \frac{\kappa}{\rho c}$ we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{k}\, \frac{\partial u}{\partial t}.$$

This equation, for example, describes the temperature, $u(x,\, y,\, t)$, in a thin plate of heat-conducing material with insulated surfaces, at any given time t.

The two-dimensional wave equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\, \frac{\partial^2 u}{\partial t^2}$$

$$\text{B.C.} \quad u(x,\, y,\, t) = 0 \qquad \text{for } (x,\, y) \text{ on the bounday}$$

$$\text{I.C.} \quad \begin{cases} u(x,\, y,\, 0) = f(x,\, y) \\ \dfrac{\partial u}{\partial t}(x,\, y,\, 0) = g(x,\, y) \end{cases}$$

This equation, for example, describes the disposition, $u(x,\, y,\, t)$, of each point of a membrane which is tretched taut over a flat frame in the xy-plane, at any given time. If the membrane line flat at time zero, then $f(x,\, y) = 0$, and if the velocity of each point of the membrane at time $t = 0$ is zero, then $g(x,\, y) = 0$.

5.3 Solution of the Two-Dimensional Heat Equation

We want to solve the initial value - boundary value problem stemming from the diffusion of the heat equation in a rectangular plate of uniform, isotropic material.

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{1}{k}\frac{\partial u}{\partial t}, && 0<x<a,\, 0<y<b,\, t>0\\
&u(x,\,0,\,t)=f_1(x),\, u(x,\,b,\,0)=f_2(x), && 0<x<a,\, t>0\\
&u(0,\,y,\,t)=g_1(y),\, u(a,\,y,\,0)=g_2(y), && 0<y<b,\, t>0\\
&u(x,\,y,\,0)=f(x,\,y), && 0<x<a,\, 0<y<b
\end{aligned}$$

The steady-state temperature, $v(x,\,y)$, is the solution of

$$\begin{aligned}
&\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0, && 0<x<a,\, 0<y<b\\
&v(x,\,0)=f_1(x),\, v(x,\,b)=f_2(x), && 0<x<a\\
&v(0,\,y)=g_1(y),\, v(a,\,y)=g_2(y), && 0<y<b
\end{aligned}$$

which we already know how to solve: Chapter 4, Section 2.

Therefore. what remains is to find the transient temperature $w(x,\,y,\,t)=u(x,\,y,\,t)-v(x,\,y)$ which satisfies

$$\begin{aligned}
&\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=\frac{1}{k}\frac{\partial w}{\partial t}, && 0<x<a,\, 0<y<b,\, t>0\\
&w(x,\,0,\,t)=w(x,\,b,\,0)=0, && 0<x<a,\, t>0\\
&w(0,\,y,\,t)=w(a,\,y,\,0)=0, && 0<y<b,\, t>0\\
&u(x,\,y,\,0)=f(x,\,y)-v(x,\,y), && 0<x<a,\, 0<y<b
\end{aligned}$$

Let $w(x,\,y,\,t)=\phi(x,\,y)h(t)$ and then apply the method of separation of variables. To solve the resulting two-dimensional eigenvalue problem in the function ϕ, let $\phi(x,\,y)=X(x)Y(y)$ and, again, apply the method of separation of variables to get two one-dimensional eigenvalue problems. Combine the solutions in a double sum using superposition principle and find the constants using the initial condition which results in a double Fourier series. *Classroom discussion!*

Remarks: 1. If f is smooth enough (all partial derivatives of large enough order exist) then the step of mathematical justification can be performed, and so $w(x, y, t)$ above will actually be a solution.

2. After learning to solve two-dimensional eigenvalue problems, we will use the Review, Identities, Formulas and Theoremshandout.

3. After learning about double Fourier series, we will use the Review, Identities, Formulas and Theoremshandout.

Exercises: Show the following.

1. $$\int_0^a \int_0^b \sin\tfrac{n\pi x}{a} \cos\tfrac{m\pi y}{b} \sin\tfrac{p\pi x}{a} \cos\tfrac{q\pi y}{b}\, dy\, dx = \begin{cases} \frac{ab}{2}, & \text{if } n = p \neq 0 \text{ and } m = q = 0 \\ \frac{ab}{4}, & \text{if } n = p \neq 0 \text{ and } m = q \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

2. $$\int_0^a \int_0^b \cos\tfrac{n\pi x}{a} \cos\tfrac{m\pi y}{b} \cos\tfrac{p\pi x}{a} \cos\tfrac{q\pi y}{b}\, dy\, dx = \begin{cases} ab, & \text{if } n = m = p = q = 0 \\ \frac{ab}{2}, & \text{if } n = p \neq 0 \text{ and } m = q = 0 \\ \frac{ab}{2}, & \text{if } n = p = 0 \text{ and } m = q \neq 0 \\ \frac{ab}{4}, & \text{if } n = p \neq 0 \text{ and } m = q \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

3. $$\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = -\lambda\phi(x, y) \\ \phi(0, y) = \phi(a, y) = 0 \\ \dfrac{\partial\phi}{\partial y}(x, 0) = \dfrac{\partial\phi}{\partial y}(x, b) = 0 \end{cases} \implies \begin{array}{l} \lambda = (\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2 \\ \phi(x) = \sin\frac{n\pi x}{a} \cos\frac{m\pi y}{b} \end{array} \quad \text{for } n = 1, 2, \cdots \text{ and } m = 0, 1, \cdots$$

4. $$\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = -\lambda\phi(x, y) \\ \dfrac{\partial\phi}{\partial x}(0, y) = \dfrac{\partial\phi}{\partial x}(a, y) = 0 \\ \dfrac{\partial\phi}{\partial y}(x, 0) = \dfrac{\partial\phi}{\partial y}(x, b) = 0 \end{cases} \implies \begin{array}{l} \lambda = (\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2 \\ \phi(x) = \cos\frac{n\pi x}{a} \cos\frac{m\pi y}{b} \end{array} \quad \text{for } n = 0, 1, \cdots \text{ and } m = 0, 1, \cdots$$

5. If $f(x, y) = \displaystyle\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} C_{nm} \sin\frac{n\pi x}{a} \cos\frac{m\pi y}{b}$ for $(x, y) \in (0, a) \times (0, b)$, then

$$C_{n0} = \tfrac{2}{ab}\int_0^a \int_0^b f(x, y) \sin\tfrac{n\pi x}{a}\, dy\, dx \text{ and } C_{nm} = \tfrac{4}{ab}\int_0^a \int_0^b f(x, y) \sin\tfrac{n\pi x}{a} \cos\tfrac{m\pi y}{b}\, dy\, dx$$

6. If $f(x, y) = \displaystyle\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{nm} \cos\frac{n\pi x}{a} \cos\frac{m\pi y}{b}$ for $(x, y) \in (0, a) \times (0, b)$, then

$$A_{00} = \tfrac{1}{ab}\int_0^a \int_0^b f(x, y)\, dy\, dx\,,\ A_{n0} = \tfrac{2}{ab}\int_0^a \int_0^b f(x, y) \cos\tfrac{n\pi x}{a}\, dy\, dx\,,$$

$$A_{0m} = \tfrac{2}{ab}\int_0^a \int_0^b f(x, y) \cos\tfrac{m\pi y}{a}\, dy\, dx \text{ and } A_{nm} = \tfrac{4}{ab}\int_0^a \int_0^b f(x, y) \cos\tfrac{n\pi x}{a} \cos\tfrac{m\pi y}{b}\, dy\, dx$$

5.4 Problems in Polar Coordinates

In this section we will motivate study of Bessel's equation by considering the problems of the vibration of a circular membrane (two-dimensional wave equation) and the conduction of heat in a circular plate (two-dimensional heat equation) in polar coordinates.

Wave	Heat	
$\nabla^2 u = \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}$	$\nabla^2 u = \dfrac{1}{k}\dfrac{\partial^2 u}{\partial t^2}$	, $0 < r < a,\ -\pi < \theta \le \pi,\ t > 0$
$u(a,\, \theta,\, t) = f(\theta)$	$u(a,\, \theta,\, t) = f(\theta)$	, $-\pi < \theta \le \pi,\ t > 0$
$u(r,\, \theta,\, 0) = g(r,\, \theta)$	$u(r,\, \theta,\, 0) = g(r,\, \theta)$	, $0 < r < a,\ -\pi < \theta \le \pi$
$\dfrac{\partial u}{\partial t}(r,\, \theta,\, 0) = h(r,\, \theta)$		, $0 < r < a,\ -\pi < \theta \le \pi$
$u(r,\, -\pi,\, t) = u(r,\, \pi,\, t)$	$u(r,\, -\pi,\, t) = u(r,\, \pi,\, t)$	, $0 < r < a,\ t > 0$
$\dfrac{\partial u}{\partial \theta}(r,\, -\pi,\, t) = \dfrac{\partial u}{\partial \theta}(r,\, \pi,\, t)$	$\dfrac{\partial u}{\partial \theta}(r,\, -\pi,\, t) = \dfrac{\partial u}{\partial \theta}(r,\, \pi,\, t)$	, $0 < r < a,\ t > 0$
$u(0,\, \theta,\, t)$ defined	$u(0,\, \theta,\, t)$ defined	, $-\pi < \theta \le \pi,\ t > 0$

Although it is obvious from the physical considerations that we are looking for a bounded solution, here we have stated it explicitly for $r = 0$ since we will use it mathematically.

The time-independent solution v is the solution of

$$
\begin{aligned}
&\nabla^2 v = 0, && 0 < r < a,\ -\pi < \theta \le \pi\\
&v(a,\, \theta) = f(\theta), && -\pi < \theta \le \pi\\
&v(r,\, -\pi) = v(r,\, \pi), && 0 < r < a\\
&\frac{\partial v}{\partial \theta}(r,\, -\pi) = \frac{\partial v}{\partial \theta}(r,\, \pi), && 0 < r < a\\
&v(0,\, \theta) \text{ defined}, && -\pi < \theta \le \pi
\end{aligned}
$$

for both cases. We can solve for v as we did in Chapter 4, Section 5. Therefore, what remains is to find the transient temperature $w(x,\, y,\, t) = u(x,\, y,\, t) - v(x,\, y)$ which satisfies

Wave	Heat	
$\nabla^2 w = \dfrac{1}{c^2}\dfrac{\partial^2 w}{\partial t^2}$	$\nabla^2 w = \dfrac{1}{k}\dfrac{\partial^2 w}{\partial t^2}$	, $0 < r < a,\ -\pi < \theta \le \pi,\ t > 0$
$w(a,\, \theta,\, t) = 0$	$w(a,\, \theta,\, t) = 0$	, $-\pi < \theta \le \pi,\ t > 0$
$w(r,\, \theta,\, 0) = g(r,\, \theta) - v(r,\, \theta)$	$w(r,\, \theta,\, 0) = g(r,\, \theta) - v(r,\, \theta)$	, $0 < r < a,\ -\pi < \theta \le \pi$
$\dfrac{\partial w}{\partial t}(r,\, \theta,\, 0) = h(r,\, \theta)$		, $0 < r < a,\ -\pi < \theta \le \pi$
$w(r,\, -\pi,\, t) = w(r,\, \pi,\, t)$	$w(r,\, -\pi,\, t) = w(r,\, \pi,\, t)$	, $0 < r < a,\ t > 0$
$\dfrac{\partial w}{\partial \theta}(r,\, -\pi,\, t) = \dfrac{\partial w}{\partial \theta}(r,\, \pi,\, t)$	$\dfrac{\partial w}{\partial \theta}(r,\, -\pi,\, t) = \dfrac{\partial w}{\partial \theta}(r,\, \pi,\, t)$	, $0 < r < a,\ t > 0$
$w(0,\, \theta,\, t)$ defined	$w(0,\, \theta,\, t)$ defined	, $-\pi < \theta \le \pi,\ t > 0$

Let $w(r,\, \theta,\, t) = \phi(r,\, \theta)h(t)$ and then apply the method of separation of variables. To solve the resulting two-dimensional eigenvalue problem in the function ϕ, let $\phi(r,\, \theta) = R(r)\Theta(\theta)$ and, again, apply the method of separation of variables. *Classroom discussion!*

$$r^2R''(r) + rR'(r) \pm \nu^2 r^2 R(r) = \mu^2 R(r), \quad 0 < r < a$$
$$R(0) \text{ defined}$$
$$R(a) = 0$$

5.5 Bessel's Equation

Definitions: Consider the ODE $P(x)\dfrac{d^2y}{dx^2} + Q(x)\dfrac{dy}{dx} + R(x)y = 0$ with P, Q and R are polynomials with no common factor.
1. A point (number) x_0 is called singular if $P(x_0) = 0$.
2. The singular point x_0 is called regular if $\lim\limits_{x\to x_0}(x-x_0)\dfrac{Q(x)}{P(x)}$ and $\lim\limits_{x\to x_0}(x-x_0)^2\dfrac{R(x)}{P(x)}$ are both finite.

Remark. This is not the most general definition of a singular point.

If x_0 is a regular singular point, then $(x-x_0)\dfrac{Q(x)}{P(x)}$ and $(x-x_0)^2\dfrac{R(x)}{P(x)}$ are analytic at x_0; they have convergent power series expansion at x_0. To solve an ODE near the regular singular point x_0, we look for a seris solution of the form $y(x) = (x-x_0)^r\sum\limits_{n=0}^{\infty} a_n(x-x_0)^2$ with r and a_n's to be determined.

Definition. An ODE of the form $x^2\, y''(x) + x\, y'(x) \pm \nu^2\, x^2\, y(x) = \mu^2\, y(x)$, with $\mu \geq 0$ and $\nu > 0$, is called a Bessel's equation.

If we divide both sides of this equation by x, we can write it in the form $(x\, y')' + (\pm\nu^2\, x - \frac{\mu^2}{x})y = 0$.

Definitions: 1. An ODE of the form $(x\, y')' + (\nu^2\, x - \frac{\mu^2}{x})y = 0$, with $\mu \geq 0$ and $\nu > 0$, is called a Bessel's equation of order μ and parameter ν.
2. An ODE of the form $(x\, y')' - (\nu^2\, x + \frac{\mu^2}{x})y = 0$, with $\mu \geq 0$ and $\nu > 0$, is called a Bessel's equation with purely imaginary argument of order μ and parameter ν.

It is easy to check that $x = 0$ is a regular singular point for the Bessel's equation. Now, we look for the series solution of the form $y = x^r\sum\limits_{n=0}^{\infty} a_n\, x^n$ with $a_0 \neq 0$. By plugging in y, $y' = \sum\limits_{n=0}^{\infty}(r+n)a_n\, x^{r+n-1}$ and $y'' = \sum\limits_{n=0}^{\infty}(r+n)(r+n-1)a_n\, x^{r+n-2}$ into the Bessel's equation, we can find its series solutions. *Classroom discussion!*

A second linearly independent solution y_2 can be found in several ways, including the following. *Classroom discussion!*

A second linearly independent solution is $y_2(x) = y_1(x) \displaystyle\int \frac{dx}{x\, y_1^2(x)}$, where $C = 0$ is used for the constant of integration.

Definitions: 1. $J_\mu(x) = \displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+\mu)!} \left(\frac{x}{2}\right)^{2m+\mu}$ is called the Bessel function of the first kind of order μ.

2. $Y_\mu(x)) = J_\mu(x) \displaystyle\int \frac{dx}{x\, J_\mu^2(x)}$, where $C = 0$ is used for the constant of integration, is called the Bessel function of the second kind of order μ.

Two linearly independent solutions of $(x\, y')' + (\nu^2\, x - \frac{\mu^2}{x})y = 0$ are $J_\mu(\nu\, x)$ and $Y_\mu(\nu\, x)$.

Properties of $J_\mu(x)$ and $Y_\mu(x)$

1. For x very small, but positive, $J_\mu(x) \approx \frac{1}{\mu!}(\frac{x}{2})^\mu$. Also, $J_0(0) = 1$ and $J_\mu(0) = 0$ for $\mu > 0$.

2. $Y_\mu(x) \approx \text{Constant} \times \begin{cases} \ln x, & \mu = 0 \\ x^{-\mu}, & \mu > 0 \end{cases}$, so $|Y_\mu(x)| \to \infty$ as $x \to 0$.

3. $J_\mu(x) \to 0$ and $Y_\mu(x) \to 0$ as $x \to \infty$.

4. $J_\mu(x)$ and $Y_\mu(x)$ have an infinite number of positive zeros. Also, $J_0(x)$ and $J_1(x)$ have no roots in common.

5. $\frac{d}{dx}(x^{-\mu} J_\mu(x)) = -x^{-\mu} J_{\mu+1}(x) \Longrightarrow \int x^{-\mu} J_{\mu+1}(x)\, dx = -x^{-\mu} J_\mu(x) + C$ and $\frac{d}{dx}(x^{\mu+1} J_{\mu+1}(x)) = x^{\mu+1} J_\mu(x) \Longrightarrow \int x^{\mu+1} J_\mu(x)\, dx = x^{\mu+1} J_{\mu+1}(x) + C$.

6. $J_0'(x) = -J_1(x) \Longrightarrow \int J_1(x)\, dx = -J_0(x) + C$ and $\frac{d}{dx}(xJ_1(x)) = xJ_0(x) \Longrightarrow \int xJ_0(x)\, dx = xJ_1(x) + C$.

7. Suppose $0 < \alpha_1 < \alpha_2 < \cdots$ are positive zeros of $J_\mu(x)$. Then

$$\int_0^a J_\mu(\frac{\alpha_n x}{a}) J_\mu(\frac{\alpha_m x}{a})\, x\, dx = \begin{cases} 0, & n \neq m \\ \frac{a^2}{2} J_{\mu+1}^2(\alpha_m) & n = m \end{cases}.$$

8. Suppose $0 < \beta_1 < \beta_2 < \cdots$ are positive zeros of $J_1(x)$ (or $J_0'(x)$). Then
$$\int_0^a J_0(\frac{\beta_n x}{a}) J_0(\frac{\beta_m x}{a})\, x\, dx = 0 \text{ for } n \neq m.$$

9. Suppose $0 < \beta_1 < \beta_2 < \cdots$ are positive zeros of $J_0(x)$. Then
$$\int_0^a J_1(\frac{\beta_n x}{a}) J_1(\frac{\beta_m x}{a})\, x\, dx = 0 \text{ for } n \neq m.$$

10. PUT GRAPHS HERE!

11. Convergence Theorems - Suppose $f(x)$ is sectionally smooth on the interval $(0, a)$ and x is any point on that interval. Then the following generalized Fourier series hold.

For $0 < \alpha_1 < \alpha_2 < \cdots$ positive zeros of $J_0(x)$,
$$\sum_{n=1}^{\infty} a_n J_0(\frac{\alpha_n x}{a}) = \frac{1}{2}(f(x^-) + f(x^+)) \text{ where } a_n = \frac{2}{a^2 J_1^2(\alpha_n)} \int_0^a f(x) J_0(\frac{\alpha_n x}{a})\, x\, dx.$$

For $0 < \alpha_1 < \alpha_2 < \cdots$ positive zeros of $J_\mu(x)$,
$$\sum_{n=1}^{\infty} a_n J_\mu(\frac{\alpha_n x}{a}) = \frac{1}{2}(f(x^-) + f(x^+)) \text{ where } a_n = \frac{2}{a^2 J_{\mu+1}^2(\alpha_n)} \int_0^a f(x) J_\mu(\frac{\alpha_n x}{a})\, x\, dx.$$

For $0 < \beta_1 < \beta_2 < \cdots$ positive zeros of $J_1(x)$ (or $J_0'(x)$),
$$a_0 + \sum_{n=1}^{\infty} a_n J_0(\frac{\beta_n x}{a}) = \frac{1}{2}(f(x^-) + f(x^+)) \text{ where } a_0 = \frac{2}{a^2} \int_0^a f(x)\, x\, dx \text{ and } a_n = \frac{\int_0^a f(x) J_0(\frac{\beta_n x}{a})\, x\, dx}{\int_0^a J_0^2(\frac{\beta_n x}{a})\, x\, dx}.$$

For $0 < \beta_1 < \beta_2 < \cdots$ positive zeros of $J_\mu'(x)$ with $\mu > 0$,
$$\sum_{n=1}^{\infty} a_n J_\mu(\frac{\beta_n x}{a}) = \frac{1}{2}(f(x^-) + f(x^+)) \text{ where } a_n = \frac{\int_0^a f(x) J_\mu(\frac{\beta_n x}{a})\, x\, dx}{\int_0^a J_\mu^2(\frac{\beta_n x}{a})\, x\, dx}.$$

Now, consider the case "$-\nu^2$" and μ still a nonnegative integer. Two linearly independent solutions of $(x\, y')' - (\nu^2 x + \frac{\mu^2}{x})y = 0$ are $I_\mu(\nu\, x)$ and $K_\mu(\nu\, x)$, which are defined below.

Definitions: 1. $I_\mu(x) = \displaystyle\sum_{m=0}^{\infty} \frac{1}{m!\,(m+\mu)!} \left(\frac{x}{2}\right)^{2m+\mu}$ is called the modified Bessel function of the first kind of order μ.

2. $K_\mu(x)) = I_\mu(x) \displaystyle\int \frac{dx}{x\, I_\mu^2(x)}$, where $C = 0$ is used for the constant of integration, is called the modified Bessel function of the second kind of order μ.

A Few Facts About $I_\mu(x)$ and $K_\mu(x)$

1. As $x \to 0$, $|K_\mu(x)| \to \infty$, $I_0(0) = 1$ and $I_\mu(0) = 0$ for $\mu > 0$.
2. $I_\mu(x)$ and $I_0'(x)$ have no real-valued zeros.
3. As $x \to \infty$, $K_\mu(x) \to 0$ and $I_\mu(x) \to \infty$ for $\mu \geq 0$.
4. PUT GRAPHS HERE!

Remarks: 1. All of the above hold even if μ is not a nonnegative integer. In that case, $(m+\mu)!$ must be replaced by $\Gamma(m+\mu+1)$ where $\Gamma(p) = \displaystyle\int_0^\infty t^{p-1}e^{-t}\,dt$.

2. $J_{\frac{1}{2}}(x) = \left(\dfrac{2}{\pi x}\right)^{\frac{1}{2}} \sin x$ and $I_{\frac{1}{2}}(x) = \left(\dfrac{2}{\pi x}\right)^{\frac{1}{2}} \operatorname{sech} x$

Exercises: 1. Show that for $\overline{m} \neq m$, $\displaystyle\int_0^a J_z(\tfrac{\alpha_m r}{a})J_z(\tfrac{\alpha_{\overline{m}} r}{a})\,r\,dr = 0$ where $0 < \alpha_1 < \alpha_2 < \cdots$ are zeros of $J_z(x)$ and $z \geq 0$.

2. Show that for $\overline{m} \neq m$, $\displaystyle\int_0^a J_z(\tfrac{\beta_m r}{a})J_z(\tfrac{\beta_{\overline{m}} r}{a})\,r\,dr = 0$ where $0 < \beta_1 < \beta_2 < \cdots$ are zeros of $J_z'(x)$ and $z \geq 0$.

3. Show that the eigenvalues of the eigenvalue problem $\begin{cases} x^2\dfrac{d^2\phi}{dx^2} + x\dfrac{d\phi}{dx} + (\lambda x^2 - n^2)\phi = 0 \\ \phi(0) \text{ bounded} \\ \phi(a) = 0 \end{cases}$ are negative, where $n = 0,\, 1,\, \cdots$.

4. Suppose $0 < \alpha_1 < \alpha_2 < \cdots$ are zeros of $J_z(x)$, with $z \geq 0$, and $f(r) = \displaystyle\sum_{n=1}^\infty a_n J_z(\frac{\alpha_n r}{a})$ for $0 < r < a$.

Show that $a_n = \dfrac{\displaystyle\int_0^a f(r)J_z(\tfrac{\alpha_n r}{a})r\,dr}{\displaystyle\int_0^a J_z^2(\tfrac{\alpha_n r}{a})r\,dr}$.

5. Solve the eigenvalue problem $\begin{cases} \dfrac{d}{dr}(r^2\dfrac{df}{dr}) + (\lambda\, r^2 - n(n+1))f = 0, \quad 0 < r < a \\ f(0) \text{ defined} \\ f(a) = 0 \end{cases}$ by using the substitution $h(r) = r^{\frac{1}{2}}f(r)$ or $f(r) = r^{-\frac{1}{2}}h(r)$, where $n = 0,\, 1,\, \cdots$.

5.6 Temperature in a Cylinder

Suppose that the temperature $u(r,\, \theta,\, t)$ in a cylinder of radius a satisfies the following problem.

$$
\begin{array}{ll}
\nabla^2 u = \dfrac{1}{k}\dfrac{\partial u}{\partial t}, & 0 < r < a,\ -\pi < \theta \le \pi,\ t > 0 \\
u(a,\, \theta,\, t) = 0, & -\pi < \theta \le \pi,\ t > 0 \\
u(r,\, \theta,\, 0) = f(r), & 0 < r < a,\ -\pi < \theta \le \pi \\
u(0,\, \theta,\, t) \text{ defined}, & -\pi < \theta \le \pi,\ t > 0
\end{array}
$$

We first should mention that if $f(r) \equiv 0$, then $u \equiv 0$ is the solution. Assuming $f(r) \not\equiv 0$, we can simplify this problem by showing that u is independent of θ. Let $u(r,\, \theta,\, t) = R(r)\Theta(\theta)h(t)$ and apply the boundary condition to show $u(r,\, \theta,\, t) = R(r)h(t)$. *Classroom discussion!*

Remark. In general, the solution will be independent of a variable, if all initial and boundary conditions are independent of that variable.

Thus our problem is reduced to finding $u(r,\, t)$ for which

$$
\begin{array}{ll}
\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} = \dfrac{1}{k}\dfrac{\partial u}{\partial t}, & 0 < r < a,\ t > 0 \\
u(a,\, t) = 0, & t > 0 \\
u(r,\, 0) = f(r), & 0 < r < a \\
u(0,\, t) \text{ defined}, & t > 0
\end{array}
$$

Let $u(r,\, t) = R(r)h(t)$, plug in to the PDE and boundary conditions to get the following problems. *Classroom discussion!*

$$
\begin{array}{ll}
(r\,R')' + \lambda\, r\, R = 0, \quad 0 < r < a & \\
R(a) = 0, & \text{and} \quad \dfrac{T'(t)}{h(t)} = -\lambda\, k \\
R(0) \text{ defined}, &
\end{array}
$$

Obviously, $h(t) = C\, e^{-\lambda k t}$. We can solve for R several different ways. 1. Consider the three cases: $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$. 2. Use one of the exercises in the last section to just consider the case $\lambda > 0$. 3. Finally, we could just use the Review, Identities, Formulas and Theoremshandout. The choice will depend on the wording of the problem. *Classroom discussion!*

In any case, we will get $\lambda = \nu_n^2 = (\frac{\alpha_n}{a})^2$, $R_n(r) = J_0(\nu_n r) = J_0(\frac{\alpha_n r}{a})$ where $0 < \alpha_1 < \alpha_2 < \cdots$ are zeros of $J_0(x)$ and $T_n(t) = e^{-\nu_n^2 kt}$. By the superposition principle, $u(r,\, t) = \displaystyle\sum_{n=1}^{\infty} a_n J_0(\frac{\alpha_n r}{a}) e^{-\frac{\alpha_n^2 kt}{a^2}}$. Find a_n's using the initial condition.

$$u(r,\, 0) = \sum_{n=1}^{\infty} a_n J_0(\frac{\alpha_n r}{a}) = f(r),\ 0 < r < a.$$

Using an earlier result, assuming f is continuous and sectionally smooth, we get

$$a_n = \frac{\displaystyle\int_0^a f(r) J_0(\frac{\alpha_n r}{a}) r\, dr}{\displaystyle\int_0^a J_0^2(\frac{\alpha_n r}{a}) r\, dr} = \frac{2}{a^2 J_1^2(\alpha_n)} \int_0^a f(r) J_0(\frac{\alpha_n r}{a}) r\, dr.$$

5.7 Vibration of a Circular Membrane

We want to solve the problem of describing the displacement of a circular membrane that is fixed at its edges. If the initial conditions are independent of θ (B.C. is already independent of θ), then solution will be independent of θ. This is the case we will solve.

$$\begin{array}{ll}
\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} = \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}, & 0 < r < a,\ t > 0 \\
u(a,\, t) = 0, & t > 0 \\
u(r,\, 0) = f(r), & 0 < r < a \\
\dfrac{\partial u}{\partial t}(r,\, 0) = g(r), & 0 < r < a \\
u(0,\, t) \text{ defined}, & t > 0
\end{array}$$

Let $u(r,\, t) = R(r)h(t)$, plug in to the PDE and boundary conditions to get the following problems. *Classroom discussion!*

$$\begin{array}{ll}
(r\,R')' + \lambda\, r\, R = 0, \quad 0 < r < a & \\
R(a) = 0, & \text{and} \quad T''(t) + \lambda\, c^2 h(t) = 0 \\
R(0) \text{ defined}, &
\end{array}$$

We have learned that $\lambda = \nu_n^2 = (\frac{\alpha_n}{a})^2$, $R_n(r) = J_0(\nu_n r) = J_0(\frac{\alpha_n r}{a})$ where $0 < \alpha_1 < \alpha_2 < \cdots$ are zeros of $J_0(x)$. The general solution of $T''(t) - (\frac{\alpha_n}{a})^2\, c^2 h(t) = 0$ is $T_n(t) = c_1 \cos(\frac{\alpha_n ct}{a}) + c_2 \sin(\frac{\alpha_n ct}{a})$.

By the superposition principle, $u(r,\, t) = \displaystyle\sum_{n=1}^{\infty} [a_n \cos(\frac{\alpha_n ct}{a}) + b_n \sin(\frac{\alpha_n ct}{a})] J_0(\frac{\alpha_n r}{a})$. Assuming we can differentiate term-by-term with respect to t, $\frac{\partial u}{\partial t}(r,\, t) = \sum_{n=1}^{\infty}[-\frac{\alpha_n c}{a} a_n \sin(\frac{\alpha_n ct}{a}) + \frac{\alpha_n c}{a} b_n \cos(\frac{\alpha_n ct}{a})] J_0(\frac{\alpha_n r}{a})$.

Find the constants a_n and b_n using the initial conditions. *Classroom discussion!*

Therefore, $u(r,\, t) = \sum_{n=1}^{\infty}[a_n \cos(\frac{\alpha_n ct}{a}) + b_n \sin(\frac{\alpha_n ct}{a})] J_0(\frac{\alpha_n r}{a})$ with

$a_n = \frac{2}{a^2 J_1^2(\alpha_n)} \int_0^a f(r) J_0(\frac{\alpha_n r}{a}) r\, dr$ and $b_n = \frac{2}{a\, \alpha_n\, c\, J_1^2(\alpha_n)} \int_0^a f(r) J_0(\frac{\alpha_n r}{a}) r\, dr$.

5.9 Spherical Coordinates; Legendre Polynomials

$x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \sin\theta$
$z = \rho \cos\phi$
$\rho = \sqrt{x^2 + y^2 + z^2}$
$-\pi \le \theta \le \pi$, $0 \le \phi \le \pi$, $\rho \ge 0$

PUT GRAPH HERE

We want to write $\nabla^2 u$ in spherical coordinates. Using the chain rule we will find the partial derivatives of ρ, ϕ and θ with respect to each of the variables x, y and z and use them to find the first and second partial derivatives of u with respect to each of the variables x, y and z.

$\frac{\partial \rho}{\partial x} = \frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{1}{2}}\, 2x = \cdots = \sin\phi\cos\theta$. Similarly, $\frac{\partial \rho}{\partial y} = \sin\phi\sin\theta$ and $\frac{\partial \rho}{\partial z} = \cos\phi$.

$\frac{\partial}{\partial z}(z = \rho\cos\phi) \Longrightarrow \cdots \Longrightarrow \frac{\partial \phi}{\partial z} = -\frac{\sin\phi}{\rho}$. Similarly, $\frac{\partial \phi}{\partial x} = \frac{\cos\theta\cos\phi}{\rho}$ (since $\frac{\partial z}{\partial x} = 0$) and $\frac{\partial \phi}{\partial y} = \frac{\sin\theta\cos\phi}{\rho}$.

$\frac{\partial}{\partial x}(x = \rho\sin\phi\cos\theta) \Longrightarrow \cdots \Longrightarrow \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{\rho\sin\phi}$, $\frac{\partial}{\partial y}(y = \rho\sin\phi\sin\theta) \Longrightarrow \cdots \Longrightarrow \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{\rho\sin\phi}$ and $\frac{\partial \theta}{\partial z} = 0$ since θ does not depend on z.

Classroom discussion!

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial x} = \cos\theta\sin\phi\frac{\partial u}{\partial \rho} - \frac{\sin\theta}{\rho\sin\phi}\frac{\partial u}{\partial \theta} + \frac{\cos\theta\cos\phi}{\rho}\frac{\partial u}{\partial \phi}$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial u}{\partial x}) = \frac{\partial}{\partial \rho}(\frac{\partial u}{\partial x})\frac{\partial \rho}{\partial x} + \frac{\partial}{\partial \theta}(\frac{\partial u}{\partial x})\frac{\partial \theta}{\partial x} + \frac{\partial}{\partial \phi}(\frac{\partial u}{\partial x})\frac{\partial \phi}{\partial x} = \cdots = \cos^2\theta\sin^2\phi\frac{\partial^2 u}{\partial \rho^2} + 2\frac{\sin\theta\cos\theta}{\rho^2\sin^2\phi}\frac{\partial u}{\partial \theta} - 2\frac{\sin\theta\cos\theta}{\rho}\frac{\partial^2 u}{\partial \rho\partial \theta} + \frac{\sin^2\theta\cos\phi - 2\cos^2\theta\sin^2\phi\cos\phi}{\rho^2\sin\phi}\frac{\partial u}{\partial \phi} + 2\frac{\cos^2\theta\sin\phi\cos\phi}{\rho}\frac{\partial^2 u}{\partial \rho\partial \phi} + \frac{\sin^2\theta + \cos^2\theta\cos^2\phi}{\rho}\frac{\partial u}{\partial \rho} + \frac{\sin^2\theta}{\rho^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} - 2\frac{\sin\theta\cos\theta\cos\phi}{\rho^2\sin\phi}\frac{\partial^2 u}{\partial \theta\partial \phi} + \frac{\cos^2\theta\cos^2\phi}{\rho^2}\frac{\partial^2 u}{\partial \phi^2}$

Classroom discussion!

Similarly, we can derive the following.

$\frac{\partial u}{\partial y} = \sin\theta\sin\phi\frac{\partial u}{\partial \rho} + \frac{\cos\theta}{\rho\sin\phi}\frac{\partial u}{\partial \theta} + \frac{\sin\theta\cos\phi}{\rho}\frac{\partial u}{\partial \phi}$

$\frac{\partial u}{\partial z} = \cos\phi\frac{\partial u}{\partial \rho} + -\frac{\sin\phi}{\rho}\frac{\partial u}{\partial \phi}$

$\frac{\partial^2 u}{\partial y^2} = \sin^2\theta\sin^2\phi\frac{\partial^2 u}{\partial \rho^2} - 2\frac{\sin\theta\cos\theta}{\rho^2\sin^2\phi}\frac{\partial u}{\partial \theta} + 2\frac{\sin\theta\cos\theta}{\rho}\frac{\partial^2 u}{\partial \rho\partial \theta} + \frac{-2\sin^2\theta\sin^2\phi\cos\phi + \cos^2\theta\cos\phi}{\rho^2\sin\phi}\frac{\partial u}{\partial \phi} + 2\frac{\sin^2\theta\sin\phi\cos\phi}{\rho}\frac{\partial^2 u}{\partial \rho\partial \phi} + \frac{\sin^2\theta\cos^2\phi + \cos^2\theta}{\rho}\frac{\partial u}{\partial \rho} + \frac{\cos^2\theta}{\rho^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} + 2\frac{\sin\theta\cos\theta\cos\phi}{\rho^2\sin\phi}\frac{\partial^2 u}{\partial \theta\partial \phi} + \frac{\sin^2\theta\cos^2\phi}{\rho^2}\frac{\partial^2 u}{\partial \phi^2}$

$\frac{\partial^2 u}{\partial z^2} = \cos^2\phi\frac{\partial^2 u}{\partial \rho^2} + 2\frac{\sin\phi\cos\phi}{\rho^2}\frac{\partial u}{\partial \phi} - 2\frac{\sin\phi\cos\phi}{\rho}\frac{\partial^2 u}{\partial \rho\partial \phi} + \frac{\sin^2\phi}{\rho}\frac{\partial u}{\partial \rho} + \frac{\sin^2\phi}{\rho^2}\frac{\partial^2 u}{\partial \phi^2}$

Classroom discussion!

After substituting in $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ and simplifying, we will get

$$\nabla^2 u = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial u}{\partial \phi}\right) + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}.$$

Classroom discussion!

Now, consider the ODE

$$\frac{d}{d\phi}\left(\sin\phi \frac{dg}{d\phi}\right) + \left(\mu \sin\phi - \frac{m^2}{\sin\phi}\right) g(\phi) = 0,\ m = 0,\ 1,\ \cdots.$$

Let $s = \cos\phi$. Then $\frac{ds}{d\phi} = \sin\phi$, $\frac{dg}{d\phi} = \frac{dg}{ds}\frac{ds}{d\phi} = -\sin\phi \frac{dg}{ds}$ and $\frac{d}{d\phi}(\sin\phi \frac{dg}{d\phi}) = \cdots = -2\sin\phi\cos\phi \frac{dg}{ds} + \sin^3\phi \frac{d^2 g}{ds^2}$. Substituting these into the ODE and simplifying, we get

$$\frac{d}{ds}\left[(1 - s^2)\frac{dg}{ds}\right] + \left[\mu - \frac{m^2}{1 - s^2}\right] g(s) = 0.$$

Classroom discussion!

5.9.1 Legendre's Equation and Polynomials

Definition. An ODE of the form $\dfrac{d}{ds}\left[(1-s^2)\dfrac{dg}{ds}\right] + \left[n(n+1) - \dfrac{m^2}{1-s^2}\right] g(s) = 0$, with $m = 0, 1, \cdots$ and $m \le n = 0, 1, \cdots$, is called a Legendre's equation.

$s = 0$ is an ordinary point for the Legendre's equation. So, we can look for the series solution of the form $g = \sum_{k=0}^{\infty} a_k\, s^k$. By plugging in g, $g' = \sum_{k=1}^{\infty} k\, a_k\, s^{k-1}$ and $g'' = \sum_{k=2}^{\infty} k(k-1)\, a_k\, s^{k-2}$ into the Legendre's equation, we can find its series solutions. *Classroom discussion!*

Definitions: 1. $P_n(s) = \sum_{l=0}^{L} (-1)^l \frac{(2n-2l)!}{l!\,(n-l)!\,(n-2l)!\,2^n} s^{n-2l}$ where $L = \frac{n}{2}$ or $\frac{n-1}{2}$, for n even or odd, respectively, is called the Legendre polynomial (of degree n).
2. $P_n^m(s) =$ TO BE COMPLETED ...

Properties of (Associated) Legendre's Polynomials

1. $P_n(s) = \dfrac{1}{n!\,2^n} \dfrac{d^n}{ds^n}(s^2-1)^n$ (Rodrigues' formula). Also, $P_n(-1) = (-1)^n$ and $P_n(1) = 1$.
2. $P_0(s) = 1$, $P_1(s) = s$, $P_2(s) = \frac{1}{2}(3s^2-1)$, $P_3(s) = \frac{1}{2}(5s^3-3s)$ and $P_4(s) = \frac{1}{8}(35s^4 - 30s^2 + 3)$. $P_n(s)$ is a polynomial of degree n.
3. PUT GRAPHS HERE!

4. $(2n+1)\,s\,P_n(s) = (n+1)P_{n+1}(s) + nP_{n-1}(s)$ and $(2n+1)P_n(s) = P'_{n+1}(s) - P'_{n-1}(s)$.
5. $\displaystyle\int_{-1}^{1} P_n(s)P_{\overline{n}}(s)\,ds = \begin{cases} 0, & \overline{n} \neq n \\ \frac{2}{2n+1}, & \overline{n} = n \end{cases}$ and $\displaystyle\int_0^{\pi} P_n(\cos\phi)P_{\overline{n}}(\cos\phi)\sin\phi\,d\phi = \begin{cases} 0, & \overline{n} \neq n \\ \frac{2}{2n+1}, & \overline{n} = n \end{cases}$.
6. $P_n^m(s) = (s^2-1)^{\frac{m}{2}} \dfrac{d^m}{ds^m} P_n(s)$ with $n \geq m > 0$. Since $P_n(s)$ is a polynomial of degree n, $\frac{d^m}{ds^m}P_n(s) = 0$ for $m > n$. This is the reason for the requirement $n \geq m$.
7. $P_1^1(s) = (s^2-1)^{\frac{1}{2}}$, $P_2^1(s) = 3s(s^2-1)^{\frac{1}{2}}$, $P_2^2(s) = 3s(s^2-1)$, $P_3^1(s) = \frac{3}{2}(5s^2-1)(s^2-1)^{\frac{1}{2}}$, $P_3^2(s) = 15s(s^2-1)$ and $P_3^3(s) = 15(s^2-1)^{\frac{3}{2}}$.
8. For $m > 0$, $\displaystyle\int_{-1}^{1} P_n^m(s)P_{\overline{n}}^m(s)\,ds = \begin{cases} 0, & \overline{n} \neq n \\ \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}, & \overline{n} = n \geq m \end{cases}$ and

$$\int_0^{\pi} P_n^m(\cos\phi)P_{\overline{n}}^m(\cos\phi)\sin\phi\,d\phi = \begin{cases} 0, & \overline{n} \neq n \\ \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}, & \overline{n} = n \geq m \end{cases}.$$

9. $(2n+1)\,s\,P_n^m(s) = (n-m+1)P_{n+1}^m(s) + (n+m)P_{n-1}^m(s)$ and

10. Convergence Theorems - Suppose $f(s)$ is sectionally smooth on the interval $(-1,\,1)$ and s is any point on that interval. Then the following generalized Fourier series hold.

$$\sum_{n=0}^{\infty} a_n P_n(s) = \frac{1}{2}(f(s^-) + f(s^+)) \text{ where } a_n = \frac{2n+1}{2}\int_{-1}^{1} f(s)P_n(s)\,ds.$$
$$\sum_{n=m}^{\infty} a_n P_n^m(s) = \frac{1}{2}(f(s^-)+f(s^+)) \text{ where } m = 0,\,1,\,\cdots \text{ and } a_n = \frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}\int_{-1}^{1} f(s)P_n^m(s)\,ds.$$

The solution of the eigenvalue problem

$$\begin{cases} \frac{d}{ds}\left[(1-s^2)\frac{dg}{ds}\right] + \left[\mu - \frac{m^2}{1-s^2}\right] g(s) = 0, \; -1 < s < 1, \; m = 0, \, 1, \, \cdots \\ g(-1) \text{ defined} \\ g(1) = 1 \end{cases}$$

is $\mu = n(n+1)$, $g(s) = P_n^m(s)$, where $m \le n = 0, 1, \cdots$. *Classroom discussion!*

5.10 Some Applications of Legendre Polynomials

Consider the potential equation in a sphere of radius a.

$$\begin{array}{ll} \nabla^2 u = 0, & 0 < \rho < a, \; -\pi < \theta \le \pi, \; 0 < \phi < \pi \\ u(a, \theta, \phi) = F(\theta, \phi), & -\pi < \theta \le \pi, \; 0 < \phi < \pi \\ u(\rho, -\pi, \phi) = u(\rho, \pi, \phi) & 0 < \rho < a, \; 0 < \phi < \pi \\ \frac{\partial u}{\partial \theta}(\rho, -\pi, \phi) = \frac{\partial u}{\partial \theta}(\rho, \pi, \phi) & 0 < \rho < a, \; 0 < \phi < \pi \\ u(0, \theta, \phi) \text{ defined,} & -\pi < \theta \le \pi, \; 0 < \phi < \pi \\ u(\rho, \theta, 0) \text{ and } u(\rho, \theta, \pi) \text{ defined,} & 0 < \rho < a, \; -\pi < \theta \le \pi \end{array}$$

Let $u(\rho, \theta, \phi) = f(\rho)q(\theta)g(\phi)$, plug in to the PDE and boundary conditions to get the following problems. *Classroom discussion!*

$$\begin{array}{ll} q''(\theta) = -\nu q(\theta), & -\pi < \theta \le \pi \\ q(-\pi) = q(\pi) & \\ q'(-\pi) = q'(\pi) & \end{array}, \quad \begin{array}{l} \frac{d}{d\phi}(\sin\phi \frac{dg}{d\phi}) + (\mu \sin\phi - \frac{\nu}{\sin\phi})g = 0, \quad 0 < \phi < \pi \\ g(0) \text{ and } g(\pi) \text{ defined} \end{array} \quad \text{and}$$

$$\frac{d}{d\rho}(\rho^2\frac{df}{d\rho}) - \mu f = 0, \quad 0 < \rho < a$$
$f(0)$ defined

We can show that $\nu = m^2$, $m = 0, 1, \cdots$, $q(\theta) = \cos m\theta, \sin m\theta$; $\mu = n(n+1)$, $m \leq n = 0, 1, \cdots$, $g(\phi) = P_n^m(\cos\phi)$; $f(\rho) = \rho^n$. *Classroom discussion!*

Thus, the product solutions are

$$u(\rho, \theta, \phi) = \rho^n \left\{ \begin{array}{c} \cos m\theta \\ \sin m\theta \end{array} \right\} P_n^m(\cos\phi),\ m = 0, 1, \cdots, m \leq n = 0, 1, \cdots.$$

By the superposition principle,

$$u(\rho, \theta, \phi) = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} A_{mn}\,\rho^n\,\cos m\theta\, P_n^m(\cos\phi) + \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} B_{mn}\,\rho^n\,\sin m\theta\, P_n^m(\cos\phi).$$

Find the constants using the boundary condition $u(a, \theta, \phi) = F(\theta, \phi)$,

$$\sum_{m=0}^{\infty}\sum_{n=m}^{\infty} A_{mn}\,a^n\,\cos m\theta\, P_n^m(\cos\phi) + \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} B_{mn}\,a^n\,\sin m\theta\, P_n^m(\cos\phi) = F(\theta, \phi).$$

Classroom discussion!

$$A_{0n} = \tfrac{2n+1}{4\pi a^2}\int_0^\pi \int_{-\pi}^\pi F(\theta,\, \phi) P_n(\cos\phi)\sin\phi\, d\theta\, d\phi,$$
$$A_{mn} = \tfrac{2n+1}{2\pi a^2}\tfrac{(n-m)!}{(n+m)!}\int_0^\pi \int_{-\pi}^\pi F(\theta,\, \phi)\cos m\theta P_n^m(\cos\phi)\sin\phi\, d\theta\, d\phi \text{ and}$$
$$B_{mn} = \tfrac{2n+1}{2\pi a^2}\tfrac{(n-m)!}{(n+m)!}\int_0^\pi \int_{-\pi}^\pi F(\theta,\, \phi)\sin m\theta P_n^m(\cos\phi)\sin\phi\, d\theta\, d\phi$$

Now, consider the wave equation on a sphere of radius a.

$$\begin{array}{ll}
\nabla^2 u = \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}, & 0<\rho<a,\, -\pi<\theta\le\pi,\, 0<\phi<\pi,\, t>0 \\
u(a,\,\theta,\,\phi,\,t) = 0, & -\pi<\theta\le\pi,\, 0<\phi<\pi,\, t>0 \\
u(\rho,\,\theta,\,\phi,\,0) = F(\rho,\,\theta,\,\phi), & 0<\rho<a, \pi<\theta\le\pi,\, 0<\phi<\pi \\
\dfrac{\partial u}{\partial t}(\rho,\,\theta,\,\phi,\,0) = 0, & 0<\rho<a, \pi<\theta\le\pi,\, 0<\phi<\pi \\
u(\rho,\,-\pi,\,\phi,\,t) = u(\rho,\,\pi,\,\phi,\,t) & 0<\rho<a,\, 0<\phi<\pi,\, t>0 \\
\dfrac{\partial u}{\partial\theta}(\rho,\,-\pi,\,\phi,\,t) = \dfrac{\partial u}{\partial\theta}(\rho,\,\pi,\,\phi,\,t) & 0<\rho<a,\, 0<\phi<\pi,\, t>0 \\
u(0,\,\theta,\,\phi,\,t) \text{ defined}, & -\pi<\theta\le\pi,\, 0<\phi<\pi,\, t>0 \\
u(\rho,\,\theta,\,0,\,t) \text{ and } u(\rho,\,\theta,\,\pi,\,t) \text{ defined}, & 0<\rho<a,\, -\pi<\theta\le\pi,\, t>0
\end{array}$$

Let $u(\rho,\,\theta,\,\phi,\,t) = w(\rho,\,\theta,\,\phi)h(t)$, plug in to the PDE and boundary conditions to get the following problems. *Classroom discussion!*

$$\begin{array}{l}
T''(t) = -\lambda\, c^2\, h(t), \quad t>0 \\
T'(0) = 0
\end{array} \quad \text{and}$$

$$\begin{array}{ll}
\nabla^2 w = -\lambda w, & 0<\rho<a,\, -\pi<\theta\le\pi,\, 0<\phi<\pi \\
w(a,\,\theta,\,\phi) = 0, & -\pi<\theta\le\pi,\, 0<\phi<\pi,\, t>0 \\
w(\rho,\,-\pi,\,\phi) = w(\rho,\,\pi,\,\phi) & 0<\rho<a,\, 0<\phi<\pi \\
\dfrac{\partial u}{\partial\theta}(\rho,\,-\pi,\,\phi) = \dfrac{\partial u}{\partial\theta}(\rho,\,\pi,\,\phi) & 0<\rho<a,\, 0<\phi<\pi \\
w(0,\,\theta,\,\phi) \text{ defined}, & -\pi<\theta\le\pi,\, 0<\phi<\pi \\
w(\rho,\,\theta,\,0) \text{ and } w(\rho,\,\theta,\,\pi) \text{ defined}, & 0<\rho<a,\, -\pi<\theta\le\pi
\end{array}$$

Let $w(\rho,\,\theta,\,\phi) = f(\rho)q(\theta)g(\phi)$, plug in to the PDE and boundary conditions to get the following problems. *Classroom discussion!*

$$\begin{aligned} &q''(\theta) = -\nu q(\theta), \quad -\pi < \theta \le \pi \\ &q(-\pi) = q(\pi) \\ &q'(-\pi) = q'(\pi) \end{aligned} \quad , \quad \begin{aligned} &\frac{d}{d\phi}(\sin\phi \frac{dg}{d\phi}) + (\mu \sin\phi - \frac{\nu}{\sin\phi})g = 0, \quad 0 < \phi < \pi \\ &g(0) \text{ and } g(\pi) \text{ defined} \end{aligned} \quad \text{and}$$

$$\begin{aligned} &\frac{d}{d\rho}(\rho^2 \frac{df}{d\rho}) + (\lambda \rho^2 - \mu)f = 0, \quad 0 < \rho < a \\ &f(0) \text{ defined} \\ &f(a) = 0 \end{aligned}$$

The solution of the first two eigenvalue problems are $\nu = m^2$, $m = 0, 1, \cdots$, $q(\theta) = \cos m\theta, \sin m\theta$; $\mu = n(n+1)$, $m \le n = 0, 1, \cdots$, $g(\phi) = P_n^m(\cos\phi)$. For the third problem, we make the substitution $h(\rho) = \rho^{\frac{1}{2}} f(\rho)$ or $f(\rho) = \rho^{-\frac{1}{2}} h(\rho)$ resulting in the following problem. *Classroom discussion!*

$$\begin{aligned} &\frac{d}{d\rho}(\rho \frac{dh}{d\rho}) + (\lambda \rho - \frac{(n+\frac{1}{2})^2}{\rho})h = 0, \quad 0 < \rho < a \\ &\rho^{-\frac{1}{2}} h(\rho) \text{ defined}, \qquad \text{as } \rho \to 0 \\ &h(a) = 0 \end{aligned}$$

The solution of this eigenvalue problem is $\lambda = (\frac{\alpha_k}{a})^2$, where $0 < \alpha_1 < \alpha_2 < \cdots$ are zeros of $J_{n+\frac{1}{2}}(x)$

and $h(\rho) = J_{n+\frac{1}{2}}(\frac{\alpha_k \rho}{a})$. So $f(\rho) = \rho^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\frac{\alpha_k \rho}{a})$ and the solution of $T''(t) = \lambda c^2 h(t)$, $T'(0) = 0$ is $h(t) = \cos \frac{\alpha_k c t}{a}$, $k = 1, 2, \cdots$. *Classroom discussion!*

Thus, the product solutions are

$$u(\rho, \theta, \phi, t) = \rho^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\tfrac{\alpha_k \rho}{a}) \left\{ \begin{array}{c} \cos m\theta \\ \sin m\theta \end{array} \right\} P_n^m(\cos\phi) \cos \tfrac{\alpha_k c t}{a} \text{ for}$$

$k = 1, 2, \cdots, m = 0, 1, \cdots$ and $m \le n = 0, 1, \cdots$.

By the superposition principle,

$$u(\rho, \theta, \phi, t) = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \sum_{k=1}^{\infty} A_{mnk}\, \rho^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\frac{\alpha_k \rho}{a}) \cos m\theta\, P_n^m(\cos\phi) \cos \frac{\alpha_k c t}{a} +$$
$$\sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \sum_{k=1}^{\infty} B_{mnk}\, \rho^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\frac{\alpha_k \rho}{a}) \sin m\theta\, P_n^m(\cos\phi) \cos \frac{\alpha_k c t}{a}.$$

Remark. In the second summation, m starts with 1, not zero, since $\sin m\theta = 0$ for $m = 0$.

Find the constants using the initial condition $u(\rho, \theta, \phi, 0) = F(\rho, \theta, \phi)$,

$$\sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \sum_{k=1}^{\infty} A_{mnk}\, \rho^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\frac{\alpha_k \rho}{a}) \cos m\theta\, P_n^m(\cos\phi) +$$
$$\sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \sum_{k=1}^{\infty} B_{mnk}\, \rho^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\frac{\alpha_k \rho}{a}) \sin m\theta\, P_n^m(\cos\phi) = F(\rho, \theta, \phi).$$

Classroom discussion!

$$A_{0nk} = \frac{2n+1}{2\pi a^2 J^2_{n+\frac{3}{2}}(\alpha_k)} \int_0^a \int_0^\pi \int_{-\pi}^\pi F(\rho,\, \theta,\, \phi) J_{n+\frac{1}{2}}(\tfrac{\alpha_k \rho}{a}) P_n(\cos\phi) \rho^{\frac{3}{2}} \sin\phi \, d\theta \, d\phi \, d\rho,$$

$$A_{mnk} = \frac{2n+1}{\pi a^2 J^2_{n+\frac{3}{2}}(\alpha_k)} \frac{(n-m)!}{(n+m)!} \int_0^a \int_0^\pi \int_{-\pi}^\pi F(\rho,\, \theta,\, \phi) J_{n+\frac{1}{2}}(\tfrac{\alpha_k \rho}{a}) \cos m\theta P_n^m(\cos\phi) \rho^{\frac{3}{2}} \sin\phi \, d\theta \, d\phi \, d\rho \text{ and}$$

$$B_{mnk} = \frac{2n+1}{\pi a^2 J^2_{n+\frac{3}{2}}(\alpha_k)} \frac{(n-m)!}{(n+m)!} \int_0^a \int_0^\pi \int_{-\pi}^\pi F(\rho,\, \theta,\, \phi) J_{n+\frac{1}{2}}(\tfrac{\alpha_k \rho}{a}) \sin m\theta P_n^m(\cos\phi) \rho^{\frac{3}{2}} \sin\phi \, d\theta \, d\phi \, d\rho$$

Remark. For a special case of heat equation on a sphere, see your textbook.